

Carleman estimates for the time-fractional advection-diffusion equations and applications

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Abstract In this article, we prove Carleman estimates for the generalized time-fractional advection-diffusion equations by considering the fractional derivative as perturbation for the first order time-derivative. As a direct application of the Carleman estimates, we show a conditional stability of a lateral Cauchy problem for the time-fractional advection-diffusion equation, and we also investigate the stability of an inverse source problem.

Keywords time-fractional advection-diffusion equation, Carleman estimate, lateral Cauchy problem, inverse source problem

AMS Subject Classifications 35R11, 35R30, 35B35

1 Introduction and main results

Recently, the position of the advection-diffusion equations (ADE) as models in a range of problems in analyzing mass transport has been challenged by more and more experiment data. For example, numerous field experiments for the solute transport in highly heterogeneous media demonstrate that solute concentration profiles exhibited anomalous non-Fickian growth rates, skewness and long-tailed profile (See e.g., [2] and [12]), which are poorly characterized by the conventional mass transport equations based on Fick's law. To more accurately interpret these effects, the non-Fickian diffusion model has been proposed to mass transport model, say, time-fractional diffusion equation (FDE):

$$\partial_t u(x, t) + \gamma \partial_t^\alpha u(x, t) = \Delta u(x, t), \quad (x, t) \in \mathbb{R}^n \times (0, \infty), \quad (1)$$

where for $\alpha \in (0, 1)$, by ∂_t^α we denote the Caputo derivative with respect to temporal variable $t > 0$:

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dg(\tau)}{d\tau} d\tau,$$

where $\Gamma(\cdot)$ is the usual Gamma function. See, e.g., [9] and [14] for the properties of the Caputo derivative, and see, e.g., [5], [16] and the references therein for the FDEs.

Introducing the time-fractional derivatives of arbitrary order into the equation of mass transport for a heterogeneous medium achieved great successes, for example, it is shown to be an efficient model for describing some anomalous diffusion processes in the highly heterogeneous media by [6] in which the authors pointed out that diffusion equation with time-fractional derivative was well-performed in describing the long-tailed profile of a particle diffuses in a highly heterogeneous medium, and by [13] where the theoretical fractional calculus on FDEs shows that there holds the non-Fick's law in the anomalous diffusion. We also refer to [16] in which the MADE site mobile tritium mass decline is well modeled by the equation (1) with the time-fractional derivative of order $\alpha = 0.33$.

In this paper, assuming $0 < \alpha_\ell < \dots < \alpha_1 < 1$, we consider a generalized time-fractional advection-diffusion equation (FADE)

$$(Lu)(x, t) \equiv \partial_t u + \sum_{j=1}^{\ell} q_j(x, t) \partial_t^{\alpha_j} u - \sum_{i,j=1}^n a_{ij}(x, t) \partial_i \partial_j u + \sum_{i=1}^n b_i(x, t) \partial_i u + c(x, t) u = F, \quad (2)$$

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where (x, t) in $\mathbb{R}^n \times (0, \infty)$. The above equation (2) has the spatially and temporally variable coefficients, and such kind of equations simulate the advection diffusion, which is more general than that in [5] and [16], and so can be regarded as more feasible model equation than symmetric fractional diffusion equations in modeling diffusion in heterogeneous media. Here in this article, we study the stability for a lateral Cauchy problem and the stability for an inverse source problem for this equation, and the stability is a fundamental theoretical subject. To the best knowledge of the authors, except the special case that $\alpha = 1/2$ which is discussed in [8], the stability results for both of the lateral Cauchy problem and inverse source problem of the equation (2) were not yet established. Here it should be mentioned that in the general order case, the transform argument and Fourier methods used in the above mentioned references [5] and [16] can not work any more because of the non-symmetry of the system and t -dependent coefficients, and it is very complicated to follow the treatment used in [8] even for $\alpha = 1/3$. One of the reasons is that in this case one needs to establish a Carleman estimate for parabolic operators of order 6 in the space variables, which will cause a huge amount of computation. However, from the shape of the equation (2) we regard the lower fractional order term as a perturbation of the first order time-derivative, which enable one to derive the Carleman estimate for the equation (2) in the framework of the Carleman estimate for the parabolic equations, and then consider the stability of the lateral Cauchy problem and inverse source problem.

To this end, we start from fixing some general settings and notations. Let $T > 0$ be fixed constant and $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 1$, with sufficiently smooth boundary $\partial\Omega$, for example, of C^2 -class. We set $Q := \Omega \times (0, T)$. Assume that $a_{ij} = a_{ji} \in C^1(\overline{Q})$, $1 \leq i, j \leq n$, satisfies that

$$\rho \sum_{j=1}^n \xi_j^2 \leq \sum_{j,k=1}^n a_{jk}(x, t) \xi_j \xi_k, \quad (x, t) \in \overline{Q}, \quad \xi \in \mathbb{R}^n,$$

where $\rho > 0$ is a constant independent of x, t, ξ . We set $\partial_{\nu_A} u = \sum_{i,j=1}^n a_{ij} \nu_i \partial_j u$ where (ν_1, \dots, ν_n) denotes the unit outwards normal vector to the boundary $\partial\Omega$. Let $L^2(\Omega)$ and $H^{k,\ell}(Q)$ ($k \geq 0, \ell \geq 0$) denote Sobolev spaces (See, e.g., [1] and [18]). Similar to Theorem 5.1 in [18], for arbitrary non-empty relatively open sub-boundary $\Gamma \subset \partial\Omega$, we choose a bounded domain Ω_1 with sufficiently smooth boundary such that

$$\Omega \subsetneq \Omega_1, \quad \overline{\Gamma} = \overline{\partial\Omega \cap \Omega_1}, \quad \partial\Omega \setminus \Gamma \subset \partial\Omega_1. \quad (3)$$

We then apply Lemma 4.1 in [18] to find a function $d \in C^2(\overline{\Omega_1})$ satisfying

$$d > 0 \text{ in } \Omega, \quad |\nabla d| > 0 \text{ on } \overline{\Omega}, \quad d = 0 \text{ on } \partial\Omega \setminus \Gamma. \quad (4)$$

Now let us first consider the equation (2) with $\alpha_1 < \frac{1}{2}$ (we call the corresponding diffusion as sub-diffusion), we set the weight function φ_1 as follows

$$\varphi_1(x, t) = e^{\lambda \psi_1(x, t)}, \quad \psi_1(x, t) = d(x) - \beta t^{2-2\alpha_1}, \quad \forall \lambda \geq 0, \quad x \in \overline{\Omega_1}, \quad t \geq 0,$$

where $\beta > 0$ is a positive constant which will be chosen later, and we have the following Carleman type estimate for the equation (2)

Theorem 1. *We assume $\alpha_1 < \frac{1}{2}$ and $q_i, b_j, c \in L^\infty(Q)$ ($i = 1, \dots, \ell, j = 1, \dots, n$) in the equation (2), and we let $\Sigma_0 = \overline{\Omega} \times \{0\}$ and $D \subset Q$ be bounded domain whose boundary ∂D is composed of a finite number of smooth surfaces. Then there exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \geq \lambda_0$, we can choose a constant $s_0(\lambda) > 0$ satisfying: there exist constants $C = C(s_0, \lambda_0) > 0$ and $C(\lambda) > 0$ such that*

$$\begin{aligned} & \int_D \left\{ \frac{1}{s\varphi_1} |\partial_t u|^2 + s\lambda^2 \varphi_1 |\nabla u|^2 + s^3 \lambda^4 \varphi_1^3 u^2 \right\} e^{2s\varphi_1} dx dt \\ & \leq C \int_D |\tilde{L}u|^2 e^{2s\varphi_1} dx dt + C(\lambda) e^{C(\lambda)s} \int_{\partial D} (|\nabla u|^2 + u^2) d\Sigma + C(\lambda) e^{C(\lambda)s} \int_{\partial D \setminus \Sigma_0} |\partial_t u|^2 d\Sigma \end{aligned} \quad (5)$$

for all $s \geq s_0$ and all $u \in H^{2,1}(D)$, where $\tilde{L} := L - \sum_{j=1}^\ell q_j(x, t) \partial_t^{\alpha_j}$.

From the above Carleman estimate, similar to the argument used in [18], we further have the conditional stability for the lateral Cauchy problem for the equation (2). However, a bit different from the results in [18], here due to the choice of the weight function in the derivation of the Carleman estimate in Theorem 1, we can only prove that the continuous dependency of the solution with respect to initial values, boundary values and source terms in the case of $\alpha_1 \in (0, \frac{1}{2})$, say,

Theorem 2. We assume $\alpha_1 < \frac{1}{2}$ and $q_i, b_j, c \in L^\infty(Q)$ ($i = 1, \dots, \ell, j = 1, \dots, n$) in the equation (2). Let $\Gamma \subset \partial\Omega$ be an arbitrary non-empty relatively open sub-boundary of $\partial\Omega$. For any bounded domain Ω_0 such that $\overline{\Omega_0} \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega \subsetneq \Gamma$ is a non-empty open subset of $\partial\Omega$, we can choose a sufficiently small $\varepsilon = \varepsilon(T, \Omega_0) > 0$ such that

$$\|u\|_{H^{1,1}(\Omega_0 \times (0, \varepsilon))} \leq C \|u\|_{H^{1,1}(Q)}^{1-\theta} \mathcal{D}^\theta, \quad (6)$$

where $\mathcal{D} := \|u(\cdot, 0)\|_{H^1(\Omega)} + \|F\|_{L^2(Q)} + \|u\|_{H^1(\Gamma \times (0, T))} + \|\partial_{\nu_A} u\|_{L^2(\Gamma \times (0, T))}$, and the constants $C > 0$ and $\theta \in (0, 1)$ may depend on T , the choice of Ω_0 and the coefficients of the equation (2).

The above arguments used for deriving Theorem 2 cannot work anymore for the general fractional order $\alpha_1 \geq \frac{1}{2}$ (the corresponding diffusion is called as sup-diffusion), but for some fractional orders, it is expected that the method of Carleman estimate still works. As a partially affirmative answer, we focus on deriving the Carleman estimate for the case of rational fractional order α_1 which is smaller than $\frac{3}{4}$, i.e. $\alpha_1 \leq \frac{3}{4}$ in the form of $\frac{m}{k}$, $m, k \in \mathbb{N}$. For this, we consider the following equation

$$(Lu)(x, t) \equiv \partial_t u + q(x) \partial_t^\alpha u - \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j u + \sum_{i=1}^n b_i(x) \partial_i u + c(x) u = F \quad (7)$$

in the case of $\alpha \in (0, \frac{3}{4}]$, where the coefficients satisfy: $b_i \in L^\infty(\Omega)$, $i = 1, \dots, n$, $c \in L^\infty(\Omega)$ and the source term F is assumed to be smooth enough. To this end, we first introduce the Riemann-Liouville fractional derivative of order $\alpha \in [m-1, m)$ with $m = 1, 2, \dots$ which is usually defined by

$$D_t^\alpha h(t) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-r)^{-\alpha+m-1} h(r) dr \quad (8)$$

and the Riemann-Liouville fractional integral $D_t^{-\alpha}$ of order $\alpha \in (0, 1)$ which is defined by

$$D_t^{-\alpha} h(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} h(r) dr. \quad (9)$$

We choose the weight function φ_2 :

$$\varphi_2(x, t) = e^{\lambda \psi_2(x, t)}, \quad \psi_2(x, t) = d(x) - \beta(t - t_0)^2 + c_0, \quad (10)$$

where $t_0 \in (0, T)$ and $\beta > 0$ are to be fixed later and $c_0 := \max\{\beta t_0^2, \beta(T - t_0)^2\}$ guarantees the non-negativeness of the function ψ_2 . In needs of applications, we establish a special Carleman estimate with a cut-off function $\chi_0 \in C^\infty(\mathbb{R}^{n+1})$ which is defined by

$$\chi_0(x, t) = \begin{cases} 1, & (x, t) \in D_0, \\ 0, & \text{outside of } D, \end{cases}$$

where D is an arbitrary subdomain of $Q_1 := \Omega_1 \times (0, T)$ and $D_0 \subsetneq D$. Now we are ready to state the Carleman estimate.

Theorem 3. For any rational number $\alpha = \frac{m}{k} \leq \frac{3}{4}$, $m, k \in \mathbb{N}$, we assume that $D_t^{\frac{j}{k}} F \in L^2(Q)$ for $j = j_1, \dots, j_k$ where $j_l := -\frac{k}{2} + \frac{(-1)^{k-1}}{4} + l$, $l = 1, \dots, k$. Then there exist constants $\hat{s} \geq 1$ and $C > 0$ such that

$$\begin{aligned} & \int_Q \chi_0^2 \left(\sum_{i,l=1}^n (s\varphi_2)^{\frac{4}{k}j_l-1} |\partial_i \partial_l u|^2 + (s\varphi_2)^{\frac{4}{k}j_1+1} |\nabla u|^2 + \sum_{j=j_1}^{j_k+k} (s\varphi_2)^{-\frac{4}{k}(j-j_1)+3} |D_t^{\frac{j}{k}} u|^2 \right) e^{2s\varphi_2} dx dt \\ & \leq C \int_Q \chi_0^2 \left(\sum_{j=j_1}^{j_k} (s\varphi_2)^{-\frac{4}{k}(j-j_1)} |D_t^{\frac{j}{k}} F|^2 \right) e^{2s\varphi_2} dx dt + Low + Bdy \end{aligned}$$

for all $s \geq \hat{s}$, large fixed $\lambda \geq 1$ and all u smooth enough satisfying (7) and $u(x, 0) = 0$ for $\forall x \in \Omega$, where Low and Bdy are defined as follows

$$Low := Cs \int_Q \left(|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \chi_0|^2 \right) \left(\sum_{j=j_1}^{j_k} (|\nabla (D_t^{\frac{j}{k}} u)|^2 + |D_t^{\frac{j}{k}} u|^2) \right) e^{2s\varphi_2} dx dt, \quad (11)$$

$$Bdy := Ce^{Cs} \int_{\partial Q} (|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + |\chi_0|^2) \left(\sum_{j=j_1}^{j_k} (|\nabla_{x,t} (D_t^{\frac{j}{k}} u)|^2 + |D_t^{\frac{j}{k}} u|^2) \right) dS dt. \quad (12)$$

We notice that the above Carleman estimate with the regular weight function φ_2 requires the homogeneous initial condition, which possibly roots in the memory effect of time-fractional derivatives. We refer to the publications [3] and [17] for the Carleman estimate for the diffusion equations with a half order time-fractional derivative where the homogeneous initial value is also necessary for deriving the Carleman estimates. Moreover, it should be mentioned here that our method can work for multi-term case where the fractional order α_j , $j = 1, \dots, l$ are all rational numbers and the largest one α_1 is smaller than $\frac{3}{4}$.

As an application of Theorem 3, we can easily derive the stability for the lateral Cauchy problem for the equation (7). For simplicity, we only state the theorem in the critical case: $\alpha = \frac{3}{4}$. The results for the case of rational order less than $\frac{3}{4}$ can be established by using the similar argument.

Theorem 4. *For any small $\epsilon > 0$ and any bounded domain Ω_0 satisfying $\overline{\Omega_0} \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega$ be a non-empty open subset of $\partial\Omega$ and $\partial\Omega_0 \cap \partial\Omega \subsetneq \Gamma$, there exist constants $C > 0$ and $\theta \in (0, 1)$ such that*

$$\|u\|_{H^{2,1}(\Omega_0 \times (\epsilon, T-\epsilon))} \leq C\mathcal{D} + CM^{1-\theta}\mathcal{D}^\theta$$

for u satisfying equation (7) with $\alpha = \frac{3}{4}$ and $u(x, 0) = 0$ for $\forall x \in \Omega$. Here M and \mathcal{D} are defined as follows

$$M := \|D_t^{\frac{1}{2}} u\|_{H^{1,0}(Q)},$$

$$\mathcal{D} := \sum_{j=-1}^2 \|D_t^{\frac{j}{4}} F\|_{L^2(Q)} + \|D_t^{\frac{1}{2}} u\|_{H^1(\Gamma \times (0, T))} + \|\partial_\nu (D_t^{\frac{1}{2}} u)\|_{L^2(\Gamma \times (0, T))}.$$

Remark 1.1. *Here the generic constant C and constant θ depend on the choice of Ω_0 , ϵ and the coefficients of the equation (7).*

Now on the basis of the above Carleman estimate, let us turn to considering another application: inverse source problem for the equation (7) where the source term is in the form of $F(x, t) = R(x, t)f(x)$. Here again we consider the critical case:

$$(Lu)(x, t) = \partial_t u + q(x)\partial_t^{\frac{3}{4}} u - \Delta u + B(x) \cdot \nabla u + c(x)u = R(x, t)f(x) \quad \text{in } Q, \quad (13)$$

where $B(x) := (b_1(x), \dots, b_n(x))$.

Problem 1 (Inverse source problem). *Fix an observation time $t_0 \in (0, T)$. We intend to determine the spatially varying factor f for given R by measuring the data on some sub-boundary and the value of the solution u at $t = t_0$.*

The measurements are as the same type as that in the case of a heat equation (See, e.g., [18] for a similar inverse source problem to heat equation). In our problem, we deal with a parabolic equation with some lower-order time-fractional derivative. The idea is to put the fractional derivative term into source term and give some suitable estimates. We have the following conditional stability result on a level set $\Omega_\epsilon := \{x \in \Omega : d(x) > \epsilon\}$ for any $\epsilon > 0$.

Theorem 5. *Assume that $R \in L^\infty(Q)$ with condition $R(\cdot, 0) = 0$ in Ω and R satisfies*

$$R(\cdot, t_0) \neq 0 \quad \text{on } \overline{\Omega}, \quad D_t^{\frac{1}{2}} R \in L^\infty(Q). \quad (14)$$

Then for any $\epsilon > 0$ there exist constants $C > 0$ and $\theta \in (0, 1)$ such that

$$\|f\|_{L^2(\Omega_{4\epsilon})} \leq C\mathcal{D} + CM^{1-\theta}\mathcal{D}^\theta$$

for all u smooth enough and satisfying the equation (13) with

$$u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 \quad \text{in } \Omega. \quad (15)$$

Here by M and \mathcal{D} we denote a priori bound and measurements as follows

$$\begin{aligned} M &:= \|f\|_{L^2(\Omega)} + \left\| D_t^{\frac{3}{2}} u \right\|_{H^{1,0}(Q)}, \\ \mathcal{D} &:= \|u(x, t_0)\|_{H^2(\Omega)} + \left\| D_t^{\frac{3}{2}} u \right\|_{H^1(\Gamma \times (0, T))} + \left\| D_t^{\frac{3}{2}} (\partial_\nu u) \right\|_{L^2(\Gamma \times (0, T))}, \end{aligned}$$

where the generic constant C and constant θ depend on ϵ and the coefficients in the equation (13).

To the best knowledge of the authors, most of the existing literatures are focused on the uniqueness of the inverse problems for the time-fractional diffusion equation, see, e.g., [4], [7], [10], [11], [19] and the references therein. This is a first attempt to attack the stability of the inverse source problem (Problem 1) for the time-fractional advection-diffusion equation. Moreover, due to our methods, it is necessary to assume both solution and the time derivative of the solution vanish at the initial time although the Carleman estimate Theorem 3 used for deriving Theorem 5 holds true only provided the homogeneous initial value.

The rest of this paper is organized in three sections. In Section 2, by regarding the fractional-order terms as non-homogeneous terms and applying the Carleman estimate for the parabolic equations, we will give a proof for Theorem 1 in the case of the highest fractional order is strictly less than half, and then as a direct application, the conditional stability of a lateral Cauchy problem for the equation (2) stated in Theorem 2 will be established. In Section 3, we first finish the proof for Theorem 3 with a regular weight function which is usually used dealing with the problems in the parabolic equations, and on the basis of the Carleman type estimate in Theorem 3, we will show that the solution continuously depends on Cauchy data and source term. Finally, concluding remarks are given in Section 4.

2 Carleman estimate for the sub-diffusion and its applications

In this section, we investigate the equation (2) with fractional order $\alpha_1 < \frac{1}{2}$. We point out that the equation (2) can be regarded a parabolic type equation if we regard the lower fractional order terms as new non-homogeneous terms. Therefore, it is expected to employ the Carleman estimate for the parabolic equations to derive the Carleman estimate for our equation, which is the key idea in this section. Owing to this treatment, in Section 2.1, we will give the proof of Theorem 1, while Theorem 2 will be proved as an application in Section 2.2.

2.1 Carleman estimate for the sub-diffusion

In this subsection, recalling the notations $d \in C^2(\overline{\Omega})$ and $|\nabla d| \neq 0$ on $\overline{\Omega}$, and the function $\psi_1 = d(x) - \beta t^{2-2\alpha_1}$ with $\beta > 0$, we follow the arguments on pp.9-19 of the survey paper [18] to prove the Carleman estimate (5). We use the same notations, where we must modify locally because our choice of the time dependency of ψ_1 is different.

Proof of Theorem 1. It is sufficient for us to discuss the derivation of a Carleman estimate for $L_0 = \partial_t - \sum_{i,j=1}^n a_{ij}(x, t) \partial_i \partial_j$ with the new weight function $\varphi_1 = e^{\lambda \psi_1}$. Namely

$$\begin{aligned} & \int_D \left\{ \frac{1}{s\varphi_1} |\partial_t u|^2 + s\lambda^2 \varphi_1 |\nabla u|^2 + s^3 \lambda^4 \varphi_1^3 u^2 \right\} e^{2s\varphi_1} dx dt \\ & \leq C \int_D |L_0 u|^2 e^{2s\varphi_1} dx dt + e^{C(\lambda)s} \int_{\partial D} (|\nabla u|^2 + |u|^2) dS dt + e^{C(\lambda)s} \int_{\partial D \setminus \Sigma_0} |\partial_t u|^2 dS dt \end{aligned}$$

for all $s \geq s_0$ and all $u \in H^{2,1}(D)$.

We note

$$\sigma(x, t) = \sum_{i,j=1}^n a_{ij}(x, t)(\partial_i d)\partial_j d, \quad (x, t) \in \overline{Q}$$

and

$$w(x, t) = e^{s\varphi_1(x,t)} u(x, t)$$

and

$$\begin{aligned} Pw = e^{s\varphi_1} L_0(e^{-s\varphi_1} w) &= \partial_t w - \sum_{i,j=1}^n a_{ij} \partial_j \partial_j w + 2s\lambda\varphi_1 \sum_{i,j=1}^n a_{ij} (\partial_i d) \partial_j w \\ &\quad - s^2 \lambda^2 \varphi_1^2 \sigma w + s\lambda^2 \varphi_1 \sigma w + s\lambda\varphi_1 w \sum_{i,j=1}^n a_{ij} \partial_i \partial_j d - s\lambda\varphi_1 w (\partial_t \psi_1). \end{aligned}$$

Now we introduce a new operator P_3 which is defined by

$$P_3 w := Pw + \left(s\lambda^2 \varphi_1 \sigma - s\lambda\varphi_1 \sum_{i,j=1}^n a_{ij} \partial_i \partial_j d + s\lambda\varphi_1 (\partial_t \psi_1) \right) w,$$

and we decompose P_3 into the parts P_1 and P_2 ,

$$P_3 w = P_1 w + P_2 w,$$

where

$$P_1 w = - \sum_{i,j=1}^n a_{ij} \partial_j \partial_j w - s^2 \lambda^2 \varphi_1^2 \sigma w,$$

and

$$P_2 w = \partial_t w + 2s\lambda\varphi_1 \sum_{i,j=1}^n a_{ij} (\partial_i d) \partial_j w + 2s\lambda^2 \varphi_1 \sigma w.$$

From the above notations for P, P_1, P_2, P_3 , it follows that

$$\begin{aligned} &\left\| e^{s\varphi_1} L_0 u + \left(s\lambda^2 \varphi_1 \sigma - s\lambda\varphi_1 \sum_{i,j=1}^n a_{ij} \partial_i \partial_j d + s\lambda\varphi_1 (\partial_t \psi_1) \right) w \right\|_{L^2(D)}^2 \\ &= \|P_1 w + P_2 w\|_{L^2(D)}^2 \geq 2 \int_D (P_1 w)(P_2 w) dx dt + \|P_2 w\|_{L^2(D)}^2. \end{aligned}$$

We will estimate $\int_D |P_2 w|^2 + 2(P_1 w)(P_2 w) dx dt$ from below. Firstly, we have

$$\begin{aligned} \int_D (P_1 w)(P_2 w) dx dt &= - \sum_{i,j=1}^n \int_D a_{ij} (\partial_i \partial_j w) \partial_t w dx dt - \sum_{i,j=1}^n \int_D a_{ij} (\partial_i \partial_j w) 2s\lambda\varphi_1 \sum_{k,l=1}^n a_{kl} (\partial_k d) \partial_l w dx dt \\ &\quad - \sum_{i,j=1}^n \int_D a_{ij} (\partial_i \partial_j w) 2s\lambda^2 \varphi_1 \sigma w dx dt - \int_D s^2 \lambda^2 \varphi_1^2 \sigma w \partial_t w dx dt \\ &\quad - \int_D 2s^3 \lambda^3 \varphi_1^3 \sigma w \sum_{i,j=1}^n a_{ij} (\partial_i d) \partial_j w dx dt - \int_D 2s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 dx dt =: \sum_{k=1}^6 J_k. \end{aligned} \quad (16)$$

Now, applying the integration by parts and the symmetry of $\{a_{ij}\}$: $a_{ij} = a_{ji}$, we give the estimates of J_k , $k = 1, \dots, 6$ separately.

$$J_1 = - \sum_{i,j=1}^n \int_D a_{ij} (\partial_i \partial_j w) \partial_t w dx dt$$

$$= \sum_{i,j=1}^n \int_D (\partial_i a_{ij})(\partial_j w) \partial_t w dx dt + \sum_{i,j=1}^n \int_D a_{ij}(\partial_j w) \partial_i \partial_t w dx dt - \sum_{i,j=1}^n \int_{\partial D} a_{ij}(\partial_j w) \nu_i \partial_t w d\Sigma.$$

Here and henceforth $\nu := (\nu_1, \dots, \nu_n, \nu_{n+1})$ denotes the unit normal exterior with respect to the boundary ∂D of D . In particular, ν_{n+1} is the component in the time direction. By noting $\nu_i = 0, \forall i = 1, \dots, n$ on Σ_0 , then integration by parts yields

$$\begin{aligned} J_1 &= \sum_{i,j=1}^n \int_D (\partial_i a_{ij})(\partial_j w) \partial_t w dx dt + \frac{1}{2} \sum_{i,j=1}^n \int_D a_{ij} \partial_t ((\partial_i w) \partial_j w) dx dt \\ &\quad - \sum_{i,j=1}^n \int_{\partial D \setminus \Sigma_0} a_{ij}(\partial_j w) \nu_i \partial_t w d\Sigma \\ &= \sum_{i,j=1}^n \int_D (\partial_i a_{ij})(\partial_j w) \partial_t w dx dt - \frac{1}{2} \sum_{i,j=1}^n \int_D (\partial_i a_{ij})(\partial_i w) \partial_j w dx dt \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_{\partial D} a_{ij}(\partial_i w)(\partial_j w) \nu_{n+1} d\Sigma - \sum_{i,j=1}^n \int_{\partial D \setminus \Sigma_0} a_{ij}(\partial_j w) \nu_i \partial_t w d\Sigma. \end{aligned}$$

Thus

$$\begin{aligned} |J_1| &\leq C \int_D |\nabla w| |\partial_t w| dx dt + C \int_D |\nabla w|^2 dx dt + C \int_{\partial D} |\nabla w|^2 d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\nabla w| |\partial_t w| d\Sigma \\ &\leq C \int_D |\nabla w| |\partial_t w| dx dt + C \int_D |\nabla w|^2 dx dt + C \int_{\partial D} |\nabla w|^2 d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\partial_t w|^2 d\Sigma. \end{aligned}$$

Since the Cauchy-Schwarz inequality implies that

$$|\nabla w| |\partial_t w| = s^{\frac{1}{2}} \lambda^{\frac{1}{2}} \varphi_1^{\frac{1}{2}} |\nabla w| s^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \varphi_1^{-\frac{1}{2}} |\partial_t w| \leq \frac{1}{2} s \lambda \varphi_1 |\nabla w|^2 + \frac{1}{2} \frac{1}{s \lambda \varphi_1} |\partial_t w|^2,$$

we have

$$\begin{aligned} |J_1| &\leq C \int_D \frac{1}{s \lambda \varphi_1} |\partial_t w|^2 dx dt + C \int_D s \lambda \varphi_1 |\nabla w|^2 dx dt \\ &\quad + C \int_{\partial D} |\nabla w|^2 d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\partial_t w|^2 d\Sigma. \end{aligned}$$

Next similar to the argument on pp. 12-13 in [18], we have

$$\begin{aligned} J_2 &= - \sum_{i,j=1}^n \sum_{k,l=1}^n \int_D 2s \lambda \varphi_1 a_{ij} a_{kl} (\partial_k d) (\partial_l w) \partial_i \partial_j w dx dt \\ &= 2s \lambda \int_D \sum_{i,j=1}^n \sum_{k,l=1}^n \lambda (\partial_i d) \varphi_1 a_{ij} a_{kl} (\partial_k d) (\partial_l w) \partial_j w dx dt \\ &\quad + 2s \lambda \int_D \sum_{i,j=1}^n \sum_{k,l=1}^n \varphi_1 \partial_i (a_{ij} a_{kl} \partial_k d) (\partial_l w) \partial_j w dx dt \\ &\quad + 2s \lambda \int_D \sum_{i,j=1}^n \sum_{k,l=1}^n \varphi_1 a_{ij} a_{kl} (\partial_k d) (\partial_i \partial_l w) \partial_j w dx dt \\ &\quad - 2s \lambda \int_{\partial D} \sum_{i,j=1}^n \sum_{k,l=1}^n \varphi_1 a_{ij} a_{kl} (\partial_k d) (\partial_l w) (\partial_j w) \nu_i d\Sigma. \end{aligned}$$

We have

$$(\text{first term}) = 2s\lambda^2 \int_D \varphi_1 \left| \sum_{i,j=1}^n a_{ij}(\partial_i d) \partial_j w \right|^2 dx dt \geq 0$$

and

$$\begin{aligned} (\text{third term}) &= s\lambda \int_D \varphi_1 \sum_{i,j=1}^n \sum_{k,l=1}^n a_{ij} a_{kl} (\partial_k d) \partial_l ((\partial_i w)(\partial_j w)) dx dt \\ &= s\lambda \int_{\partial D} \varphi_1 \sum_{i,j=1}^n \sum_{k,l=1}^n a_{ij} a_{kl} (\partial_k d) (\partial_i w)(\partial_j w) \nu_l d\Sigma \\ &\quad - s\lambda^2 \int_D \varphi_1 \sum_{i,j=1}^n \sigma a_{ij} (\partial_i w) \partial_j w dx dt \\ &\quad - s\lambda \int_D \varphi_1 \sum_{i,j=1}^n \sum_{k,l=1}^n \partial_l (a_{ij} a_{kl} (\partial_k d)) (\partial_i w) \partial_j w dx dt, \end{aligned}$$

which imply

$$J_2 \geq - \int_D s\lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij} (\partial_i w) \partial_j w dx dt - C \int_D s\lambda \varphi_1 |\nabla w|^2 dx dt - C \int_{\partial D} s\lambda \varphi_1 |\nabla w|^2 d\Sigma.$$

$$\begin{aligned} J_3 &= 2 \int_D s\lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij} (\partial_i w) \partial_j w dx dt + 2 \int_D s\lambda^2 \sum_{i,j=1}^n \partial_i (\varphi_1 \sigma a_{ij}) w \partial_j w dx dt \\ &\quad - 2 \int_{\partial D} s\lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij} w (\partial_j w) \nu_i d\Sigma \\ &\geq 2 \int_D s\lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij} (\partial_i w) \partial_j w dx dt - C \int_D s\lambda^3 \varphi_1 |\nabla w| |w| dx dt - C \int_{\partial D} s\lambda^2 \varphi_1 |\nabla w| |w| d\Sigma. \end{aligned}$$

By

$$s\lambda^3 \varphi_1 |\nabla w| |w| = (s\lambda^2 \varphi_1 |w|)(\lambda |\nabla w|) \leq \frac{1}{2} s^2 \lambda^4 \varphi_1^2 w^2 + \frac{1}{2} \lambda^2 |\nabla w|^2,$$

and

$$s\lambda^2 \varphi_1 |\nabla w| |w| = (s\lambda^{\frac{3}{2}} \varphi_1 |w|)(\lambda^{\frac{1}{2}} |\nabla w|) \leq \frac{1}{2} s^2 \lambda^3 \varphi_1^2 w^2 + \frac{1}{2} \lambda |\nabla w|^2,$$

we have

$$\begin{aligned} J_3 &\geq 2 \int_D s\lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij} (\partial_i w) \partial_j w dx dt - C \int_D s^2 \lambda^4 \varphi_1^2 w^2 dx dt - C \int_D \lambda^2 |\nabla w|^2 dx dt \\ &\quad - C \int_{\partial D} \lambda |\nabla w|^2 d\Sigma - C \int_{\partial D} s^2 \lambda^3 \varphi_1^2 w^2 d\Sigma. \end{aligned}$$

$$\begin{aligned} |J_4| &= \left| -\frac{1}{2} \int_D s^2 \lambda^2 \varphi_1^2 \sigma \partial_t (w^2) dx dt \right| \\ &= \left| \int_D s^2 \lambda^3 \varphi_1^2 \beta (2\alpha_1 - 2) t^{1-2\alpha_1} \sigma w^2 dx dt + \frac{1}{2} \int_D s^2 \lambda^2 \varphi_1^2 (\partial_t \sigma) w^2 dx dt - \frac{1}{2} \int_{\partial D} s^2 \lambda^2 \varphi_1^2 \sigma w^2 \nu_{n+1} d\Sigma \right| \\ &\leq C \int_D s^2 \lambda^3 \varphi_1^2 w^2 dx dt + C \int_{\partial D} s^2 \lambda^2 \varphi_1^2 w^2 d\Sigma. \end{aligned}$$

$$\begin{aligned}
J_5 &= - \int_D s^3 \lambda^3 \varphi_1^3 \sum_{i,j=1}^n \sigma a_{ij} (\partial_i d) \partial_j (w^2) dx dt \\
&= 3 \int_D s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 dx dt + \int_D s^3 \lambda^3 \varphi_1^3 \sum_{i,j=1}^n \partial_j (\sigma a_{ij} \partial_i d) w^2 dx dt \\
&\quad - \int_{\partial D} s^3 \lambda^3 \varphi_1^3 \sum_{i,j=1}^n \sigma a_{ij} (\partial_i d) w^2 \nu_j d\Sigma \\
&\geq 3 \int_D s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 dx dt - C \int_D s^3 \lambda^3 \varphi_1^3 w^2 dx dt - C \int_{\partial D} s^3 \lambda^3 \varphi_1^3 w^2 d\Sigma.
\end{aligned}$$

$$J_6 = - \int_D 2s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 dx dt.$$

By the definition of P_2 , we have

$$\begin{aligned}
\epsilon \int_D \frac{1}{s\varphi_1} |\partial_t w|^2 dx dt &= \epsilon \int_D \frac{1}{s\varphi_1} \left| P_2 w - 2s\lambda\varphi_1 \sum_{i,j=1}^n a_{ij} (\partial_i d) \partial_j w - 2s\lambda^2 \varphi_1 \sigma w \right|^2 dx dt \\
&\leq C \int_D |P_2 w|^2 dx dt + C\epsilon \int_D s\lambda^2 \varphi_1 |\nabla w|^2 dx dt + C\epsilon \int_D s\lambda^4 \varphi_1 w^2 dx dt.
\end{aligned}$$

By summing up all the above estimates for J_k , $k = 1, \dots, 6$, we find

$$\begin{aligned}
&\int_D s^3 \lambda^4 \varphi_1^3 \sigma^2 w^2 dx dt + \int_D s\lambda^2 \varphi_1 \sigma \sum_{i,j=1}^n a_{ij} (\partial_i w) \partial_j w dx dt + \left(\epsilon - \frac{C}{\lambda} \right) \int_D \frac{1}{s\varphi_1} |\partial_t w|^2 dx dt \\
&\leq C \int_D |L_0 u|^2 e^{2s\varphi_1} dx dt + C \int_D (s\lambda\varphi_1 + \epsilon s\lambda^2 \varphi_1 + \lambda^2) |\nabla w|^2 dx dt + C \int_D (s^3 \lambda^3 \varphi_1^3 + s^2 \lambda^4 \varphi_1^2) w^2 dx dt \\
&\quad + C \int_{\partial D} (s\lambda\varphi_1 |\nabla w|^2 + s^3 \lambda^3 \varphi_1^3 w^2) d\Sigma + C \int_{\partial D \setminus \Sigma_0} |\partial_t w|^2 d\Sigma,
\end{aligned}$$

which combined with the ellipticity of a_{ij} and $\sigma_0 := \inf_{(x,t) \in Q} \sigma(x,t) > 0$ yields that

$$\begin{aligned}
&\int_D s^3 \lambda^4 \varphi_1^3 \sigma_0^2 w^2 dx dt + \int_D (\sigma_0 \rho - C\epsilon) s\lambda^2 \varphi_1 |\nabla w|^2 dx dt + \left(\epsilon - \frac{C}{\lambda} \right) \int_D \frac{1}{s\varphi_1} |\partial_t w|^2 dx dt \\
&\leq C \int_D |L_0 u|^2 e^{2s\varphi_1} dx dt + C \int_D (s^3 \lambda^3 \varphi_1^3 + s^2 \lambda^4 \varphi_1^2) w^2 dx dt + C \int_D (s\lambda\varphi_1 + \lambda^2) |\nabla w|^2 dx dt \\
&\quad + \int_{\partial D} (s\lambda\varphi_1 |\nabla w|^2 + s^2 \lambda^3 \varphi_1^2 w^2) d\Sigma + \int_{\partial D \setminus \Sigma_0} |\partial_t w|^2 d\Sigma.
\end{aligned}$$

Thus choosing $\epsilon > 0$ small, and choosing λ and then s large, we can absorb terms suitably to obtain

$$\begin{aligned}
&\int_D \left\{ \frac{1}{s\varphi_1} |\partial_t w|^2 + s\lambda^2 \varphi_1 |\nabla w|^2 + s^3 \lambda^4 \varphi_1^3 w^2 \right\} dx dt \\
&\leq C \int_D |L_0 u|^2 e^{2s\varphi_1} dx dt + \int_{\partial D} (s\lambda\varphi_1 |\nabla w|^2 + s^2 \lambda^3 \varphi_1^2 w^2) d\Sigma + \int_{\partial D \setminus \Sigma_0} |\partial_t w|^2 d\Sigma.
\end{aligned}$$

Noting $w = ue^{s\varphi_1}$, we have

$$\begin{aligned}
&\int_D \left\{ \frac{1}{s\varphi_1} |\partial_t u|^2 + s\lambda^2 \varphi_1 |\nabla u|^2 + s^3 \lambda^4 \varphi_1^3 u^2 \right\} e^{2s\varphi_1} dx dt \\
&\leq C \int_D |L_0 u|^2 e^{2s\varphi_1} dx dt + C(\lambda) e^{C(\lambda)s} \int_{\partial D} (|\nabla u|^2 + u^2) d\Sigma + C(\lambda) e^{C(\lambda)s} \int_{\partial D \setminus \Sigma_0} |\partial_t u|^2 d\Sigma,
\end{aligned}$$

which completes the proof of the theorem. \square

2.2 Application to a lateral Cauchy problem for the sub-diffusion

In this section, we will give a proof of Theorem 2. To prove this, we follow the usual argument used in analyzing the lateral Cauchy problem for the parabolic equations, that is, we use the Carleman estimate derived in Section 2.1. The first problem which we have to overcome is to evaluate the fractional derivative by the first order time-derivative under some suitable norm. Namely, the following lemma holds.

Lemma 2.1. *Let $T > 0$ and $0 < \alpha \leq \alpha_1 < \frac{1}{2}$ be given constants, then the following inequality*

$$\int_{\mathcal{C}_1} |\partial_t^\alpha u|^2 e^{2s\varphi_1} dx dt \leq C \int_{\mathcal{C}_2} \frac{1}{s\lambda\varphi_1} |\partial_t u|^2 e^{2s\varphi_1} dx dt \quad (17)$$

holds true for all $u \in H^{2,1}(Q)$, where $\varphi_1 = e^{\lambda\psi_1}$ with $\psi_1(x, t) = d(x) - \beta t^{2-2\alpha_1}$, and $\mathcal{C}_i := \{(x, t); x \in \overline{\Omega}, t > 0, \varphi_1(x, t) > c_i\}$, $i = 1, 2$, and c_i are positive constants such that $c_2 < c_1$.

Proof. We choose a nonnegative function $\Phi \in L^\infty(\mathbb{R}^{n+1})$ such that $\text{supp}\Phi \subset \mathcal{C}_2$ and $\Phi \equiv 1$ in \mathcal{C}_1 . Thus we have

$$\int_{\mathcal{C}_1} |\partial_t^\alpha u|^2 e^{2s\varphi_1} dx dt = \int_{\mathcal{C}_1} \Phi(x, t) |\partial_t^\alpha u|^2 e^{2s\varphi_1} dx dt,$$

which combined with the definition of the Caputo derivative implies that

$$\int_{\mathcal{C}_1} \Phi(x, t) |\partial_t^\alpha u|^2 e^{2s\varphi_1} dx dt = \int_{\mathcal{C}_1} \left| \frac{\Phi^{\frac{1}{2}}(x, t)}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \partial_r u(x, r) dr \right|^2 e^{2s\varphi_1} dx dt.$$

Moreover, since for any fixed $x \in \Omega$, $\varphi_1(x, t)$ is decreasing with respect to the variable $t > 0$, we see that $\varphi_1(x, t) \leq \varphi_1(x, r)$ for any $x \in \Omega$ and $0 < r \leq t$, so that $(x, t) \in \mathcal{C}_1$ implies that $(x, r) \in \mathcal{C}_1$ for $0 < r \leq t$, hence that $\Phi(x, r) = 1$ if $(x, t) \in \mathcal{C}_1$ and $0 < r < t$, finally we have

$$\begin{aligned} \int_{\mathcal{C}_1} \Phi(x, t) |\partial_t^\alpha u|^2 e^{2s\varphi_1} dx dt &= \int_{\mathcal{C}_1} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \Phi^{\frac{1}{2}}(x, r) \partial_r u(x, r) dr \right|^2 e^{2s\varphi_1} dx dt \\ &\leq \int_Q \left| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} \Phi^{\frac{1}{2}}(x, r) \partial_r u(x, r) dr \right|^2 e^{2s\varphi_1} dx dt. \end{aligned}$$

Now by noting that $\varphi_1(x, t) \geq c_0$, where $c_0 > 0$ is a constant, we have

$$\partial_t \psi_1 = \beta(2\alpha_1 - 2)t^{1-2\alpha_1}, \quad \partial_t \varphi_1 = \lambda(\partial_t \psi_1) \varphi_1 = (2\alpha_1 - 2)\beta\lambda\varphi_1 t^{1-2\alpha_1}.$$

Hence

$$t^{1-2\alpha_1} e^{2s\varphi_1} = -\frac{1}{4\beta s\lambda\varphi_1(1-\alpha_1)} \partial_t (e^{2s\varphi_1}). \quad (18)$$

By the Cauchy-Schwarz inequality and (3), we have

$$\begin{aligned} &\int_0^T \left| \int_0^t (t-r)^{-\alpha} \Phi^{\frac{1}{2}}(x, r) \partial_r u(r) dr \right|^2 e^{2s\varphi_1(x, t)} dt \\ &\leq \int_0^T \left(\int_0^t (t-r)^{-2\alpha} dr \right) \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1(x, t)} dt \\ &= \frac{1}{1-2\alpha} \int_0^T t^{1-2\alpha} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1(x, t)} dt \end{aligned}$$

Moreover, since $0 < \alpha < \alpha_1$, we see that

$$\int_0^T \left| \int_0^t (t-r)^{-\alpha} \Phi^{\frac{1}{2}}(x, r) \partial_r u(r) dr \right|^2 e^{2s\varphi_1} dt \leq \frac{T^{2(\alpha_1-\alpha)}}{1-2\alpha} \int_0^T t^{1-2\alpha_1} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1} dt.$$

Now from the formula (18), integration by parts implies

$$\begin{aligned}
& \int_0^T t^{1-2\alpha_1} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1(x, t)} dt \\
&= \frac{1}{1-\alpha_1} \left(\frac{-1}{4\beta s \lambda \varphi_1} \int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1(x, t)} \Big|_{t=0}^{t=T} + \frac{1}{1-\alpha_1} \int_0^T \frac{\Phi(x, t)}{4\beta s \lambda \varphi_1} |\partial_t u|^2 e^{2s\varphi_1} dt \\
&+ \int_0^T \frac{t^{1-2\alpha_1}}{2s\varphi_1} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1} dt \\
&\leq \frac{1}{1-\alpha_1} \int_0^T \frac{\Phi(x, t)}{4\beta s \lambda \varphi_1} |\partial_t u|^2 e^{2s\varphi_1} dt + \int_0^T \frac{t^{1-2\alpha_1}}{2s\varphi_1} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1} dt.
\end{aligned}$$

The last term on the right-hand side can be absorbed into the left-hand side by choosing $s > 0$ large and we have

$$\int_0^T t^{1-2\alpha_1} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1(x, t)} dt \leq C \int_0^T \frac{\Phi(x, t)}{s\lambda\varphi_1} |\partial_t u(t)|^2 e^{2s\varphi_1} dt.$$

Thus

$$\int_Q t^{1-2\alpha_1} \left(\int_0^t \Phi(x, r) |\partial_r u(r)|^2 dr \right) e^{2s\varphi_1(x, t)} dx dt \leq C \int_Q \frac{\Phi(x, t)}{s\lambda\varphi_1} |\partial_t u(t)|^2 e^{2s\varphi_1} dx dt,$$

which combined with the fact $\text{supp}\Phi \subset \mathcal{C}_2$ implies that

$$\int_{\mathcal{C}_1} |\partial_t^\alpha u|^2 e^{2s\varphi_1} dx dt \leq C \int_{\mathcal{C}_2} \frac{1}{s\lambda\varphi_1} |\partial_t u|^2 e^{2s\varphi_1} dx dt,$$

which completes the proof of the lemma. \square

Before giving the proof of Theorem 2, we introduce some notations.

For arbitrary given domain Ω_0 such that $\overline{\Omega_0} \subset \Omega$, similar to Theorem 5.1 in [18], we will choose a suitable weight function $\psi_1(x, t) := d(x) - \beta t^{2-2\alpha_1}$. For this, we first choose a bounded domain Ω_1 with smooth boundary such that

$$\Omega \subsetneq \Omega_1, \quad \overline{\Gamma} = \overline{\partial\Omega \cap \Omega_1}, \quad \partial\Omega \setminus \Gamma \subset \partial\Omega_1.$$

We then apply Lemma 4.1 in [18] to obtain $d \in C^2(\overline{\Omega_1})$ satisfying

$$d(x) > 0, \quad x \in \Omega_1, \quad d(x) = 0, \quad x \in \partial\Omega_1, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega}.$$

Then we can choose $\beta > 0$ such that

$$\beta \left(\frac{T}{2} \right)^{2-2\alpha_1} < \|d\|_{C(\overline{\Omega_1})} < \beta T^{2-2\alpha_1}. \quad (19)$$

Moreover, since $\overline{\Omega_0} \subset \Omega_1$, we can choose a sufficiently large $N > 1$ such that

$$\Omega_0 \subset \overline{\Omega} \cap \{x \in \Omega_1; \quad d(x) > \frac{4}{N} \|d\|_{C(\overline{\Omega_1})}\}. \quad (20)$$

We set $\mu_k = \exp\{\lambda(\frac{k}{N} \|d\|_{C(\overline{\Omega_1})} - \frac{\beta(\frac{T}{2})^{2-2\alpha_1}}{N})\}$, and $D_k := \{(x, t); \quad x \in \overline{\Omega}, \quad t > 0, \quad \varphi(x, t) > \mu_k\}$, $k = 1, 2, 3, 4$. Then we can verify from (19) and (20) that

$$\Omega_0 \times (0, \frac{T}{2M}) \subset D_4 \subset D_3 \subset D_1 \subset \overline{\Omega} \times (0, T), \quad (21)$$

where $M := N^{\frac{1}{2-2\alpha_1}}$, and

$$\partial D_1 \subset \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \quad (22)$$

are valid. Here $\Sigma_0 = \{(x, 0); \quad x \in \overline{\Omega}\}$, $\Sigma_1 \subset \Gamma \times (0, T)$ and $\Sigma_2 = \{(x, t); \quad x \in \Omega, \quad t > 0, \quad \varphi(x, t) = \mu_1\}$.

Now we are ready to give the proof of our main theorem.

Proof of Theorem 2. We start from the Cauchy problem

$$\begin{cases} u(x, t) = g_0(x, t) & \text{on } \Gamma \times (0, T], \\ \partial_{\nu_A} u(x, t) = g_1(x, t) & \text{on } \Gamma \times (0, T] \end{cases}$$

for the equation (2).

Henceforth $C > 0$ denotes generic constants depending on λ , but independent of s and the choice of g_0, g_1, u . For it, we need a cut-off function because we have no data $\partial_{\nu_A} u$ on $\partial D \setminus \Gamma \times (0, T)$. Let $\chi \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \chi \leq 1$ and

$$\chi(x, t) = \begin{cases} 1, & \varphi_1(x, t) > \mu_3, \\ 0, & \varphi_1(x, t) < \mu_2. \end{cases} \quad (23)$$

Setting $v := \chi u$, $\tilde{L} := L - \sum_{j=1}^\ell q_j \partial_t^{\alpha_j}$, and then using Leibniz's formula for the differential of the product we have

$$\tilde{L}v = \chi \tilde{L}u + A_1 u = \chi Lu - \chi \sum_{j=1}^\ell q_j \partial_t^{\alpha_j} u + A_1 u = \chi F - \chi \sum_{j=1}^\ell q_j \partial_t^{\alpha_j} u + A_1 u. \quad (24)$$

Here the last term $A_1 u$ involves only the linear combination of $(\partial_t \chi)u$, $(\partial_i \partial_j \chi)u$, $(\partial_i \chi)(\partial_j u)$ and $(\partial_i \chi)u$, $i, j = 1, \dots, n$.

By (22) and (23), we see that $v = |\nabla v| = \partial_t v = 0$ on Σ_2 . Hence using the Carleman estimate in Theorem 1, from $D_3 \subset D_1$ by an argument similar to Theorem 3.2 in [18] in D_1 to (24), we find

$$\begin{aligned} & \int_{D_3} \left\{ \frac{1}{s\varphi_1} |\partial_t v|^2 + s\lambda^2 \varphi_1 |\nabla v|^2 + s^3 \lambda^4 \varphi_1^3 v^2 \right\} e^{2s\varphi_1} dx dt \\ & \leq \int_Q F^2 e^{2s\varphi_1} dx dt + C \int_{D_1} \sum_{j=1}^\ell \chi^2(x, t) |\partial_t^{\alpha_j} u|^2 e^{2s\varphi_1} dx dt + C \int_{D_1} |A_1 u|^2 e^{2s\varphi_1} dx dt \\ & \quad + e^{C(\lambda)s} \int_{\Sigma_0 \cup (\Gamma \times (0, T))} (|\nabla v|^2 + v^2) d\Sigma + e^{C(\lambda)s} \int_{\Gamma \times (0, T)} |\partial_t v|^2 dS dt. \end{aligned} \quad (25)$$

for all $s \geq s_0$ and $\lambda \geq \lambda_0$.

By (23), $A_1 u$ does not vanish only if $\mu_2 \leq \varphi(x, t) \leq \mu_3$ and so

$$\int_{D_1} |A_1 u|^2 e^{2s\varphi_1} dx dt \leq C e^{2s\mu_3} \|u\|_{H^{1,0}(Q)}^2.$$

Moreover, from (21) and Lemma 2.1, letting $\mathcal{C}_1 = \{(x, t); x \in \overline{\Omega}, t > 0, \varphi_1(x, t) > \frac{\mu_3 + \mu_4}{2}\}$ and $\mathcal{C}_2 = D_3$ in Lemma 2.1, for λ being large fixed, we conclude that the integration $\int_{\mathcal{C}_1} \sum_{j=1}^\ell \chi^2(x, t) |\partial_t^{\alpha_j} u|^2 e^{2s\varphi_1} dx dt$ can be absorbed by the left-hand side of (25), which implies

$$\begin{aligned} & \int_{D_3} \left\{ \frac{1}{s\varphi_1} |\partial_t u|^2 + s\varphi_1 |\nabla u|^2 + s^3 \varphi_1^3 u^2 \right\} e^{2s\varphi_1} dx dt \\ & \leq C e^{Cs} \|F\|_{L^2(Q)} + C e^{2s\mu_3} \|u\|_{H^{1,0}(Q)}^2 + C \sum_{j=1}^\ell \int_{D_1 \setminus \mathcal{C}_1} |\partial_t^{\alpha_j} u|^2 e^{2s\varphi_1} dx dt \\ & \quad + e^{Cs} \int_{\Sigma_0 \cup (\Gamma \times (0, T))} (|\nabla v|^2 + v^2) d\Sigma + e^{Cs} \int_{\Gamma \times (0, T)} |\partial_t v|^2 dS dt. \end{aligned}$$

By (20), we can directly verify that $\varphi_1(x, t) \leq \frac{\mu_3 + \mu_4}{2}$ in $D_1 \setminus \mathcal{C}_1$, and if $(x, t) \in \Omega_0 \times (0, \varepsilon)$, then $\varphi_1(x, t) > \mu_4$. Then combined with (21) and (22), by Hölder's inequality, we have

$$\begin{aligned} & e^{2s\mu_4} \int_0^{\frac{T}{2M}} \int_{\Omega_0} \left\{ \frac{1}{s} |\partial_t u|^2 + s |\nabla u|^2 + s^3 u^2 \right\} dx dt \\ & \leq C e^{Cs} \|F\|_{L^2(Q)} + C e^{2s\mu_3} \|u\|_{H^{1,0}(Q)}^2 + C e^{2s\frac{\mu_3 + \mu_4}{2}} \int_Q |\partial_t u|^2 dx dt \end{aligned}$$

$$+ e^{Cs} \int_{\Omega} (|\nabla v(x, 0)|^2 + v^2(x, 0)) dx + e^{Cs} \int_{\Gamma \times (0, T)} (|\partial_t v|^2 + |\nabla v|^2 + v^2) dS dt.$$

for $s \geq s_0$. Then dividing both sides by $e^{2s\mu_4}$, since

$$se^{-2s\frac{\mu_4-\mu_3}{2}} \leq Ce^{-\frac{s(\mu_4-\mu_3)}{2}}, \quad se^{-2s(\mu_4-\mu_3)} \leq Ce^{-s(\mu_4-\mu_3)} \leq Ce^{-\frac{s(\mu_4-\mu_3)}{2}},$$

by replacing C by Ce^{Cs_0} , we have

$$\|u\|_{H^{1,1}(\Omega_0 \times (0, \frac{T}{2M}))}^2 \leq Ce^{-\frac{\mu_4-\mu_3}{2}s} \|u\|_{H^{1,1}(Q)}^2 + Ce^{Cs} \mathcal{D}^2 \quad (26)$$

for all $s \geq 0$ and $u \in H^{2,1}(Q)$, where the constant $C > 0$ depends on T, Ω_0 and the coefficients of the equation (2).

Firstly, if $\mathcal{D} = 0$, letting $s \rightarrow \infty$, we conclude that $u = 0$ in $\Omega_0 \times (0, \frac{T}{2M})$, so that the conclusion of Theorem 2 holds true. Next let $\mathcal{D} \neq 0$. First let $\mathcal{D} \geq \|u\|_{H^{1,1}(Q)}$. Then (26) implies

$$\|u\|_{H^{1,1}(\Omega_0 \times (0, \frac{T}{2M}))} \leq Ce^{Cs} \mathcal{D}, \quad s \geq 0,$$

which already proves the theorem. Second let $\mathcal{D} \leq \|u\|_{H^{1,1}(Q)}$, we choose $s > 0$ minimizing the right-hand side of (26), that is

$$e^{-\frac{\mu_4-\mu_3}{2}s} \|u\|_{H^{1,1}(Q)}^2 = e^{Cs} \mathcal{D}^2.$$

By $\mathcal{D} \neq 0$, we can choose

$$s = \frac{4}{2C + \mu_4 - \mu_3} \log \frac{\|u\|_{H^{1,1}(Q)}}{\mathcal{D}} > 0.$$

Then (26) gives

$$\|u\|_{H^{1,1}(\Omega_0 \times (0, \frac{T}{2M}))} \leq 2C \|u\|_{H^{1,1}(Q)}^{1-\theta} \mathcal{D}^\theta,$$

where $\theta := \frac{\mu_4-\mu_3}{2C+\mu_4-\mu_3}$, and the constant C depends on Ω_0, T and the coefficients of the equation (2). We complete the proof of the theorem by setting $\varepsilon = \frac{T}{2M}$. \square

3 Carleman estimate for a sup-diffusion and its applications

In this section, we pay attention to the Carleman estimate for the equation (7) in the case of $\alpha = \frac{m}{k}$, say,

$$\partial_t u + q(x) \partial_t^{\frac{m}{k}} u - \Delta u + B \cdot \nabla u + cu = F \quad \text{in } Q, \quad (27)$$

and its applications to the lateral Cauchy problem and the inverse source problem. Without loss of generality, we set $a_{ij} = \delta_{ij}$, $1 \leq i, j \leq n$ here. In the following two subsections, we will give the proofs of Theorems 3, 4 and 5 respectively.

3.1 Carleman estimate for a sup-diffusion

In this subsection, we will give a proof of Theorem 3. Similar to the case of $\alpha < \frac{1}{2}$, we construct the Carleman estimate for the following parabolic type equation

$$\partial_t u - \Delta u + B \cdot \nabla u + cu = F - q(x) \partial_t^\alpha u, \quad (28)$$

whereas here we further multiply on both sides of the above equation by several Riemann-Liouville fractional derivatives in order to dealing the new source term $\partial_t^\alpha u$. We have the following details of the proof.

Proof of Theorem 3. For some technical reasons, we divide the proof into two cases:

- (i) $\alpha = \frac{m}{2k+1}$, $m, k \in \mathbb{Z}$, $k \geq 1$, $m = 1, 2, \dots, 2k$,
- (ii) $\alpha = \frac{m}{2k}$, $m, k \in \mathbb{Z}$, $k \geq 1$, $m = 1, 2, \dots, 2k - 1$.

In the first part, we consider the case (i) in which the denominator is odd. Because of the zero initial condition, the equation (28) can be rephrased as

$$\tilde{L}u := \partial_t u - \Delta u + B \cdot \nabla u + cu = F - q(x)D_t^{\frac{m}{2k+1}}u \quad \text{in } Q. \quad (29)$$

Recalling the definition of Riemann-Liouville fractional integral operator D_t^{-p} :

$$D_t^{-p}u := {}_0 D_t^{-p}u = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s) ds,$$

and we have the semigroup property for the R.-L. fractional integral operators:

$$D_t^{-p_1} D_t^{-p_2} u = D_t^{-p_1-p_2} u, \quad \forall p_1, p_2 > 0.$$

We first apply the fractional differential(or integral) operators $D_t^{\frac{j}{2k+1}}$ to the equation (29) separately for $j = -k, \dots, k$ to derive

$$D_t^{\frac{j}{2k+1}}(\partial_t u) - D_t^{\frac{j}{2k+1}}(\Delta u) + D_t^{\frac{j}{2k+1}}(B \cdot \nabla u) + D_t^{\frac{j}{2k+1}}(cu) = D_t^{\frac{j}{2k+1}}F - D_t^{\frac{j}{2k+1}}(qD_t^{\frac{m}{2k+1}}u), \quad j = -k, \dots, k.$$

Moreover, the homogeneous initial value implies that the differential operators and the R.-L. integral operator are commutable, which along with the formula $D_t^{\frac{j}{2k+1}}(D_t^{\frac{m}{2k+1}}u) = D_t^{\frac{j}{2k+1} + \frac{m}{2k+1}}u = D_t^{\frac{j+m}{2k+1}}u$ implies that

$$\tilde{L}(u_j) = \partial_t u_j - \Delta u_j + B \cdot \nabla u_j + cu_j = D_t^{\frac{j}{2k+1}}F - qD_t^{\frac{j+m}{2k+1}}u, \quad j = -k, \dots, k \quad (30)$$

where $u_j := D_t^{\frac{j}{2k+1}}u$, for all $j \in \mathbb{Z}$. Next we denote $\tilde{u}_j := \chi_0 u_j$. Then by noting that

$$\tilde{L}(\chi_0 u) - \chi_0 \tilde{L}u = (\partial_t \chi_0)u - 2\nabla \chi_0 \cdot \nabla u - (\Delta \chi_0)u + (B \cdot \nabla \chi_0)u,$$

the equations (30) can be rewritten by:

$$\tilde{L}(\tilde{u}_j) = \chi_0 D_t^{\frac{j}{2k+1}}F - (\chi_0 q)u_{j+m} - 2\nabla \chi_0 \cdot \nabla u_j + (B \cdot \nabla \chi_0 - \Delta \chi_0 + \partial_t \chi_0)u_j \quad (31)$$

for all $j = -k, \dots, k$. Now we use a Carleman estimate for parabolic type stated in the following lemma.

Lemma 3.1. *Let $F \in L^2(Q)$. Then there exist constants $\hat{\lambda} \geq 1$, $\hat{s} \geq 1$ and $C > 0$, $C(\lambda) > 0$ such that*

$$\begin{aligned} & \int_Q \left\{ s^{\tau-1} \lambda^\tau \varphi_2^{\tau-1} \left(|\partial_t u|^2 + \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) + s^{\tau+1} \lambda^{\tau+2} \varphi_2^{\tau+1} |\nabla u|^2 + s^{\tau+3} \lambda^{\tau+4} \varphi_2^{\tau+3} |u|^2 \right\} e^{2s\varphi_2} dx dt \\ & \leq C \int_Q (s\lambda\varphi_2)^\tau |F|^2 e^{2s\varphi_2} dx dt + C(\lambda) e^{C(\lambda)s} s^\tau \int_{\partial Q} (|\nabla_{x,t} u|^2 + |u|^2) dS dt \end{aligned}$$

for all $s \geq \hat{s}$, $\lambda \geq \hat{\lambda}$, $\tau \in \mathbb{Z}$ and all u smooth enough satisfying the equation:

$$\tilde{L}u = F.$$

The proof of this lemma is similar to that of Theorem 3.2 in [18]. However we need to calculate more carefully when we do the integration by parts because we have a new weight φ_2 . To specify the dependence, C and $\hat{\lambda}$ depends on $\|\nabla d\|$ while \hat{s} also depends on τ, β, T and the coefficients of the operator \tilde{L} . Furthermore, $C(\lambda)$ even depends on large parameter λ but is independent of s .

Applying the above lemma to the equations (31), we have the following Carleman estimates:

$$\begin{aligned}
& \lambda \int_Q \left\{ (s\lambda\varphi_2)^{\tau_j-1} \left(|\partial_t \tilde{u}_j|^2 + \sum_{i,l=1}^n |\partial_i \partial_l \tilde{u}_j|^2 \right) + (s\lambda\varphi_2)^{\tau_j+1} |\nabla \tilde{u}_j|^2 + (s\lambda\varphi_2)^{\tau_j+3} |\tilde{u}_j|^2 \right\} e^{2s\varphi_2} dx dt \\
& \leq C \int_Q (s\lambda\varphi_2)^{\tau_j} (|\chi_0 D_t^{\frac{j}{2k+1}} F|^2 + |\chi_0 u_{j+m}|^2) e^{2s\varphi_2} dx dt + C \int_Q (s\lambda\varphi_2)^{\tau_j} (|\tilde{L}(\chi_0 u_j) - \chi_0 \tilde{L}(u_j)|^2) e^{2s\varphi_2} dx dt \\
& \quad + C(\lambda) s^{\tau_j} e^{C(\lambda)s} \int_{\partial Q} (|\nabla_{x,t}(\chi_0 u_j)|^2 + |\chi_0 u_j|^2) dS dt
\end{aligned} \tag{32}$$

for all $\lambda \geq \lambda_j$, $s \geq s_j$ and $j = -k, \dots, k$.

Moreover, we observe that a direct calculation implies that

$$\begin{aligned}
|\partial_t \tilde{u}_j|^2 &= |(\partial_t \chi_0) u_j + \chi_0 (\partial_t u_j)|^2 \geq \frac{1}{2} |\chi_0 (\partial_t u_j)|^2 - |(\partial_t \chi_0) u_j|^2, \\
|\partial_i \partial_l \tilde{u}_j|^2 &= |\chi_0 \partial_i \partial_l u_j + (\partial_i \chi_0) (\partial_l u_j) + (\partial_l \chi_0) (\partial_i u_j) + (\partial_i \partial_l \chi_0) u_j|^2 \\
&\geq \frac{1}{2} |\chi_0 \partial_i \partial_l u_j|^2 - 3 |(\partial_i \chi_0) (\partial_l u_j)|^2 - 3 |(\partial_l \chi_0) (\partial_i u_j)|^2 - 3 |(\partial_i \partial_l \chi_0) u_j|^2, \\
|\nabla \tilde{u}_j|^2 &= |\chi_0 \nabla u_j + u_j \nabla \chi_0|^2 \geq \frac{1}{2} |\chi_0 \nabla u_j|^2 - |\nabla \chi_0|^2 |u_j|^2.
\end{aligned}$$

Thus, Carleman inequalities (32) lead to

$$\begin{aligned}
& \lambda \int_Q \chi_0^2 \left\{ (s\lambda\varphi_2)^{\tau_0-1} \left(|\partial_t u|^2 + \sum_{i,l=1}^n |\partial_i \partial_l u|^2 \right) + (s\lambda\varphi_2)^{\tau_0+1} |\nabla u|^2 + (s\lambda\varphi_2)^{\tau_0+3} |u|^2 \right\} e^{2s\varphi_2} dx dt \\
& \leq C \int_Q \chi_0^2 (s\lambda\varphi_2)^{\tau_0} (|F|^2 + |u_m|^2) e^{2s\varphi_2} dx dt + C(\lambda) s^{\tau_0} e^{C(\lambda)s} \int_{\partial Q} (|\nabla_{x,t}(\chi_0 u)|^2 + |\chi_0 u|^2) dS dt \\
& \quad + C \int_Q (s\lambda\varphi_2)^{\tau_0} \left(|\tilde{L}(\chi_0 u) - \chi_0 \tilde{L}u|^2 + (|\partial_t \chi_0|^2 + s\lambda^2 \varphi_2 |\nabla \chi_0|^2 + \sum_{i,l=1}^n |\partial_i \partial_l \chi_0|^2) |u|^2 + |\nabla \chi_0|^2 |\nabla u|^2 \right) e^{2s\varphi_2} dx dt
\end{aligned} \tag{33}$$

for all $\lambda \geq \lambda_0$, $s \geq s_0$ and

$$\begin{aligned}
& \lambda \int_Q \chi_0^2 \left\{ (s\lambda\varphi_2)^{\tau_j-1} |\partial_t u_j|^2 + (s\lambda\varphi_2)^{\tau_j+1} |\nabla u_j|^2 + (s\lambda\varphi_2)^{\tau_j+3} |u_j|^2 \right\} e^{2s\varphi_2} dx dt \\
& \leq C \int_Q \chi_0^2 (s\lambda\varphi_2)^{\tau_j} (|D_t^{\frac{j}{2k+1}} F|^2 + |u_{j+m}|^2) e^{2s\varphi_2} dx dt + C(\lambda) s^{\tau_j} e^{C(\lambda)s} \int_{\partial Q} (|\nabla_{x,t}(\chi_0 u_j)|^2 + |\chi_0 u_j|^2) dS dt \\
& \quad + C \int_Q (s\lambda\varphi_2)^{\tau_j} (|\tilde{L}(\chi_0 u_j) - \chi_0 \tilde{L}(u_j)|^2 + |\partial_t \chi_0|^2 |u_j|^2 + s\lambda^2 \varphi_2 |\nabla \chi_0|^2 |u_j|^2) e^{2s\varphi_2} dx dt
\end{aligned} \tag{34}$$

for all $\lambda \geq \lambda_j$, $s \geq s_j$ and all $j = -k, \dots, -1, 1, \dots, k$. Combining the Carleman inequalities (33)–(34), i.e., taking the summation, we obtain

$$\begin{aligned}
& \lambda \int_Q \chi_0^2 \left\{ (s\lambda\varphi_2)^{\tau_0-1} \sum_{i,l=1}^n |\partial_i \partial_l u|^2 + (s\lambda\varphi_2)^{\tau_0+1} |\nabla u|^2 + \sum_{j=-k}^k ((s\lambda\varphi_2)^{\tau_j-1} |\partial_t u_j|^2 + (s\lambda\varphi_2)^{\tau_j+3} |u_j|^2) \right\} e^{2s\varphi_2} dx dt \\
& \leq C \int_Q \chi_0^2 \sum_{j=-k}^k (s\lambda\varphi_2)^{\tau_j} |D_t^{\frac{j}{2k+1}} F|^2 e^{2s\varphi_2} dx dt + C \int_Q \chi_0^2 \sum_{j=-k}^k (s\lambda\varphi_2)^{\tau_j} |u_{j+m}|^2 e^{2s\varphi_2} dx dt + Low1 + Bdy1
\end{aligned}$$

for all $\lambda \geq \tilde{\lambda} := \max\{\lambda_j : -k \leq j \leq k\}$ and $s \geq \tilde{s} := \max\{s_j : -k \leq j \leq k\}$. Here *Low1* and *Bdy1* are the lower order terms and boundary terms determined as

$$Low1 = C \int_Q \sum_{j=-k}^k (s\lambda\varphi_2)^{\tau_j} \left(|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i,l=1}^n |\partial_i \partial_l \chi_0|^2 \right) (|\nabla u_j|^2 + s\lambda^2 \varphi_2 |u_j|^2) e^{2s\varphi_2} dx dt,$$

$$Bdy1 = C(\lambda)e^{C(\lambda)s} \int_{\partial Q} \sum_{j=-k}^k s^{\tau_j} (|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + |\chi_0|^2) (|\nabla_{x,t} u_j|^2 + |u_j|^2) dSdt.$$

Now we fix $\tau_j = -\frac{4}{2k+1}(j+k) \leq 0$, $j = -k, \dots, k$. Along with the relation

$$u_{j+2k+1} = D_t^{\frac{j+2k+1}{2k+1}} u = \partial_t (D_t^{\frac{j}{2k+1}} u) = \partial_t u_j$$

which follows after the zero initial condition, a direct calculation yields that

$$\begin{aligned} \sum_{j=-k}^k (s\lambda\varphi_2)^{\tau_j-1} |u_{j+2k+1}|^2 &= \sum_{j=-k}^k (s\lambda\varphi_2)^{-\frac{4j+6k+1}{2k+1}} |u_{j+2k+1}|^2 = \sum_{j=k+1}^{3k+1} (s\lambda\varphi_2)^{-\frac{4j-2k-3}{2k+1}} |u_j|^2, \\ \sum_{j=-k}^k (s\lambda\varphi_2)^{\tau_j+3} |u_j|^2 &= \sum_{j=-k}^k (s\lambda\varphi_2)^{-\frac{4j-2k-3}{2k+1}} |u_j|^2, \\ \sum_{j=-k}^k (s\lambda\varphi_2)^{\tau_j} |u_{j+m}|^2 &= \sum_{j=-k}^k (s\lambda\varphi_2)^{-\frac{4j+4k}{2k+1}} |u_{j+m}|^2 = \sum_{j=-k+m}^{k+m} (s\lambda\varphi_2)^{-\frac{4j+4k-4m}{2k+1}} |u_j|^2, \end{aligned}$$

which imply that

$$\begin{aligned} &\lambda \int_Q \chi_0^2 \left\{ \sum_{i,l=1}^n (s\lambda\varphi_2)^{-\frac{6k+1}{2k+1}} |\partial_i \partial_l u|^2 + (s\lambda\varphi_2)^{-\frac{2k-1}{2k+1}} |\nabla u|^2 + \sum_{j=-k}^{3k+1} (s\lambda\varphi_2)^{-\frac{4j-2k-3}{2k+1}} |u_j|^2 \right\} e^{2s\varphi_2} dxdt \\ &\leq C \int_Q \chi_0^2 \sum_{j=-k}^k (s\lambda\varphi_2)^{-\frac{4j+4k}{2k+1}} |D_t^{\frac{j}{2k+1}} F|^2 e^{2s\varphi_2} dxdt + C \int_Q \chi_0^2 \sum_{j=-k+m}^{k+m} (s\lambda\varphi_2)^{-\frac{4j+4k-4m}{2k+1}} |u_j|^2 e^{2s\varphi_2} dxdt \\ &\quad + Low1 + Bdy1 \end{aligned} \tag{35}$$

for all $\lambda \geq \tilde{\lambda}$ and all $s \geq \tilde{s}$. Since $\alpha = \frac{m}{2k+1} \in (0, \frac{3}{4}]$ implies

$$-\frac{4j-2k-3}{2k+1} \geq -\frac{4j+4k-4m}{2k+1} \quad \text{for } j = -k+m, \dots, k+m,$$

we then fix λ large so that the second terms on the RHS of (35) can be absorbed into the LHS of it ($\lambda \geq \hat{\lambda}$). Moreover, we have

$$\begin{aligned} Low1 &\leq C \int_Q \sum_{j=-k}^k \left(|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i,l=1}^n |\partial_i \partial_l \chi_0|^2 \right) (|\nabla u_j|^2 + s\varphi_2 |u_j|^2) e^{2s\varphi_2} dxdt \leq Low, \\ Bdy1 &\leq Ce^{Cs} \int_{\partial Q} \sum_{j=-k}^k (|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + |\chi_0|^2) (|\nabla_{x,t} u_j|^2 + |u_j|^2) dSdt = Bdy \end{aligned}$$

since $s, \lambda, \varphi_2 \geq 1$ and $\tau_j \leq 0$. Therefore, (35) yields

$$\begin{aligned} &\int_Q \chi_0^2 \left\{ \sum_{i,l=1}^n (s\varphi_2)^{-\frac{6k+1}{2k+1}} |\partial_i \partial_l u|^2 + (s\varphi_2)^{-\frac{2k-1}{2k+1}} |\nabla u|^2 + \sum_{j=-k}^{3k+1} (s\varphi_2)^{-\frac{4j-2k-3}{2k+1}} |D_t^{\frac{j}{2k+1}} u|^2 \right\} e^{2s\varphi_2} dxdt \\ &\leq C \int_Q \chi_0^2 \sum_{j=-k}^k (s\varphi_2)^{-\frac{4j+4k}{2k+1}} |D_t^{\frac{j}{2k+1}} F|^2 e^{2s\varphi_2} dxdt + Low + Bdy \end{aligned} \tag{36}$$

for λ large fixed and all $s \geq \tilde{s}$.

For the case (ii), we repeat the strategy of above to obtain the equations similar to (31):

$$\tilde{L}(\tilde{u}_j) = \chi_0 D_t^{\frac{j}{2k}} F - (\chi_0 q) u_{j+m} - 2\nabla \chi_0 \cdot \nabla u_j + (B \cdot \nabla \chi_0 - \Delta \chi_0 + \partial_t \chi_0) u_j \tag{37}$$

for all $j = -k+1, \dots, k$ where we denote $u_j := D_t^{\frac{j}{2k}} u$ and $\tilde{u}_j = \chi_0 u_j$ for all $j \in \mathbb{Z}$ instead. Again we apply Carleman estimate for parabolic equations (Lemma 3.1) to (36) and take the summation over j from $-k+1$ to k which reads

$$\begin{aligned} & \lambda \int_Q \chi_0^2 \left\{ (s\lambda\varphi_2)^{\tau_0-1} \sum_{i,l=1}^n |\partial_i \partial_l u|^2 + (s\lambda\varphi_2)^{\tau_0+1} |\nabla u|^2 + \sum_{j=-k+1}^k ((s\lambda\varphi_2)^{\tau_j-1} |u_{j+2k}|^2 + (s\lambda\varphi_2)^{\tau_j+3} |u_j|^2) \right\} e^{2s\varphi_2} dx dt \\ & \leq C \int_Q \chi_0^2 \sum_{j=-k+1}^k (s\lambda\varphi_2)^{\tau_j} |D_t^{\frac{j}{2k}} F|^2 e^{2s\varphi_2} dx dt + C \int_Q \chi_0^2 \sum_{j=-k+1}^k (s\lambda\varphi_2)^{\tau_j} |u_{j+m}|^2 e^{2s\varphi_2} dx dt + Low2 + Bdy2 \end{aligned} \quad (38)$$

for all $\lambda \geq \tilde{\lambda}$ and $s \geq \tilde{s}$. Here $Low2$ and $Bdy2$ are the lower order terms and boundary terms determined as

$$\begin{aligned} Low2 &= C \int_Q \sum_{j=-k+1}^k (s\lambda\varphi_2)^{\tau_j} \left(|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i,l=1}^n |\partial_i \partial_l \chi_0|^2 \right) (|\nabla u_j|^2 + s\lambda^2 \varphi_2 |u_j|^2) e^{2s\varphi_2} dx dt, \\ Bdy2 &= C(\lambda) e^{C(\lambda)s} \int_{\partial Q} \sum_{j=-k+1}^k s^{\tau_j} (|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + |\chi_0|^2) (|\nabla_{x,t} u_j|^2 + |u_j|^2) dS dt. \end{aligned}$$

For the case (ii), we choose $\tau_j = -\frac{2}{k}(j+k-1) \leq 0$, $j = -k+1, \dots, k$. By direct calculation, we obtain

$$\begin{aligned} \sum_{j=-k+1}^k (s\lambda\varphi_2)^{\tau_j-1} |u_{j+2k}|^2 &= \sum_{j=-k+1}^k (s\lambda\varphi_2)^{-\frac{2j+3k-2}{k}} |u_{j+2k}|^2 = \sum_{j=k+1}^{3k} (s\lambda\varphi_2)^{-\frac{2j-k-2}{k}} |u_j|^2, \\ \sum_{j=-k+1}^k (s\lambda\varphi_2)^{\tau_j+3} |u_j|^2 &= \sum_{j=-k+1}^k (s\lambda\varphi_2)^{-\frac{2j-k-2}{k}} |u_j|^2, \\ \sum_{j=-k+1}^k (s\lambda\varphi_2)^{\tau_j} |u_{j+m}|^2 &= \sum_{j=-k+1}^k (s\lambda\varphi_2)^{-\frac{2j+2k-2}{k}} |u_{j+m}|^2 = \sum_{j=-k+m+1}^{k+m} (s\lambda\varphi_2)^{-\frac{2j+2k-2m-2}{k}} |u_j|^2. \end{aligned}$$

Since $\alpha = \frac{m}{2k} \in (0, \frac{3}{4}]$ implies that

$$-\frac{2j-k-2}{k} \geq -\frac{2j+2k-2m-2}{k} \quad \text{for } j = -k+m+1, \dots, k+m,$$

we fix λ so large that the second terms on the RHS of (38) can be absorbed. Therefore, we have the following Carleman inequality

$$\begin{aligned} & \int_Q \chi_0^2 \left\{ \sum_{i,l=1}^n (s\varphi_2)^{-\frac{3k-2}{k}} |\partial_i \partial_l u|^2 + (s\varphi_2)^{-\frac{k-2}{k}} |\nabla u|^2 + \sum_{j=-k}^{3k} (s\varphi_2)^{-\frac{2j-k-2}{k}} |D_t^{\frac{j}{2k}} u|^2 \right\} e^{2s\varphi_2} dx dt \\ & \leq C \int_Q \chi_0^2 \sum_{j=-k+1}^k (s\varphi_2)^{-\frac{2j+2k-2}{k}} |D_t^{\frac{j}{2k}} F|^2 e^{2s\varphi_2} dx dt + Low + Bdy \end{aligned} \quad (39)$$

for λ large fixed and all $s \geq \tilde{s}$.

Theorem 3 now follows after the two Carleman inequalities (36) and (39). \square

Remark 3.1. Under the assumption $h(0) = 0$, we note that

$$\|D_t^\alpha h\|_{L^2(0,T)} \leq C \|D_t^\beta h\|_{L^2(0,T)}, \quad -1 < \alpha < \beta < 2, \quad (40)$$

which actually can be derived by using the definition of D_t^α and Young's inequality. In view of this estimate, we can also rewrite Bdy as

$$Bdy = C_1 e^{Cs} \int_{\partial Q} (|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + |\chi_0|^2) (|\nabla_{x,t} D_t^{\frac{1}{2}} u|^2 + |D_t^{\frac{1}{2}} u|^2) d\Sigma.$$

However, constant C_1 now depends on k , which is the main difficulty in dealing with the equation (7) with the time-fractional derivative of irrational order by using the density of rational numbers in \mathbb{R} . The Carleman estimate for the equation (7) with the general order time-fractional derivatives remains open.

3.2 Application to a lateral Cauchy problem for the sup-diffusion

In this subsection, we employ the Carleman estimate in Theorem 3 to investigate the conditional stability for the lateral Cauchy problem. For this, we recall the partial differential equation:

$$\partial_t u + q(x) \partial_t^{\frac{3}{4}} u - \Delta u + B \cdot \nabla u + cu = F \quad \text{in } Q \quad (41)$$

with the zero initial condition:

$$u(x, 0) = 0, \quad x \in \Omega. \quad (42)$$

We mainly follow the steps in [18] Theorem 5.1. Instead of a classical Carleman estimate for parabolic equation, we should apply our Carleman estimate established in Section 3.1.

Proof of Theorem 4. By the choice of Ω_0 , we have $\overline{\Omega_0} \subset \Omega_1$ where Ω_1 is defined in Section 1. Then we can find a sufficiently large $N > 1$ such that

$$\left\{ x \in \Omega_1 : d(x) > \frac{3}{N} \|d\|_{C(\overline{\Omega_1})} \right\} \cap \overline{\Omega} \supset \Omega_0. \quad (43)$$

Moreover we choose $\beta > 0$ such that

$$\beta \epsilon^2 \leq \|d\|_{C(\overline{\Omega_1})} \leq 2\beta \epsilon^2. \quad (44)$$

For any $t_0 \in [\sqrt{2}\epsilon, T - \sqrt{2}\epsilon]$, we set $\mu_k := \exp\{\frac{k}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta \epsilon^2}{N} + c_0\}$, $E_k := \{(x, t) \in Q_1 : \varphi_2(x, t) > \mu_k\}$ and $D_k := E_k \cap Q$, $k = 1, 2, 3$. Recall that c_0 is the same constant as that in (10). Then we can easily verify the following two facts:

- (i) $\Omega_0 \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}) \subset D_3 \subset D_2 \subset D_1 \subset \overline{\Omega} \times (t_0 - \sqrt{2}\epsilon, t_0 + \sqrt{2}\epsilon)$,
- (ii) $\partial D_1 \subset \Sigma_1 \cup \Sigma_2$, $\Sigma_1 \subset \Gamma \times (0, T)$, $\Sigma_2 = \{(x, t) \in Q : \varphi_2(x, t) = \mu_1\}$.

Next, we specify the cut-off function χ_0 in Theorem 3 with $D_0 = E_2$ and $D = E_1$. In detail, $\chi_0 \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \chi_0 \leq 1$ and

$$\chi_0(x, t) = \begin{cases} 1, & \varphi_2(x, t) > \mu_2, \\ 0, & \varphi_2(x, t) \leq \mu_1. \end{cases}$$

Thus from Theorem 3 (denominator $k = 4$ in this proof), it follows that

$$\begin{aligned} & \int_{D_1} \chi_0^2 \left(\sum_{j=-1}^6 s^{2-j} |D_t^{\frac{j}{4}} u|^2 + |\nabla u|^2 + s^{-2} \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) e^{2s\varphi_2} dx dt \\ & \leq C \int_{D_1} \chi_0^2 \left(\sum_{j=-1}^2 |D_t^{\frac{j}{4}} F|^2 \right) e^{2s\varphi_2} dx dt + Low + Bdy \end{aligned} \quad (45)$$

for all $s \geq s_0$ where Low and Bdy is the same notations as (11)-(12). In the above inequality, we have used the fact that φ_2 is upper and lower bounded by a constant which is independent of s . Since the second term Low on the RHS includes some derivatives of χ_0 , it vanishes in $\overline{E_2}$ and outside of E_1 , which implies that $\varphi_2(x, t) \leq \mu_2$ in $E_1 \setminus \overline{E_2} (\supset D_1 \setminus \overline{D_2})$ and thus reads

$$Low = Cs \int_Q \left(|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \chi_0|^2 \right) \sum_{j=-1}^2 \left(|\nabla (D_t^{\frac{j}{4}} u)|^2 + |D_t^{\frac{j}{4}} u|^2 \right) e^{2s\varphi_2} dx dt$$

$$\leq Cse^{2s\mu_2} \int_{D_1 \setminus \overline{D_2}} \sum_{j=-1}^2 \left(|\nabla(D_t^{\frac{j}{4}}u)|^2 + |D_t^{\frac{j}{4}}u|^2 \right) dxdt \leq Cse^{2s\mu_2} \|D_t^{\frac{1}{2}}u\|_{H^{1,0}(Q)}^2.$$

The last inequality above holds after the estimate (40). Furthermore, fact (ii) and the choice of χ_0 yield that $\nabla\chi_0, \partial_t\chi_0$ and χ_0 vanish on both $\Omega \times \{0, T\}$ and $(\partial\Omega \setminus \Gamma) \times (0, T)$ which indicate that

$$\begin{aligned} Bdy &= Ce^{Cs} \int_{\partial Q} (|\partial_t\chi_0|^2 + |\nabla\chi_0|^2 + |\chi_0|^2) \sum_{j=-1}^2 \left(|\nabla_{x,t}(D_t^{\frac{j}{4}}u)|^2 + |D_t^{\frac{j}{4}}u|^2 \right) dSdt \\ &\leq Ce^{Cs} \int_{\Gamma \times (0,T)} \sum_{j=-1}^2 \left(|\nabla_{x,t}(D_t^{\frac{j}{4}}u)|^2 + |D_t^{\frac{j}{4}}u|^2 \right) dSdt \\ &\leq Ce^{Cs} \left(\|\nabla_{x,t}(D_t^{\frac{1}{2}}u)\|_{L^2(\Gamma \times (0,T))}^2 + \|D_t^{\frac{1}{2}}u\|_{L^2(\Gamma \times (0,T))}^2 \right). \end{aligned}$$

The last inequality again comes from (40). Thus, we obtain

$$\begin{aligned} RHS \text{ of (45)} &\leq Ce^{Cs} \sum_{j=-1}^2 \int_{D_1} |D_t^{\frac{j}{4}}F|^2 dxdt + Cse^{2s\mu_2} \|D_t^{\frac{1}{2}}u\|_{H^{1,0}(Q)}^2 \\ &\quad + Ce^{Cs} \left(\|\nabla_{x,t}(D_t^{\frac{1}{2}}u)\|_{L^2(\Gamma \times (0,T))}^2 + \|D_t^{\frac{1}{2}}u\|_{L^2(\Gamma \times (0,T))}^2 \right) \\ &\leq Cse^{2s\mu_2} M^2 + Ce^{Cs} \mathcal{D}^2 \end{aligned}$$

where M and \mathcal{D} are defined in Theorem 4. On the other hand, the inclusion $\Omega_0 \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}) \subset D_3 \subset D_2$ and $\chi_0 = 1$ in D_2 imply that

$$\begin{aligned} LHS \text{ of (45)} &= \int_{D_2} \left(\sum_{j=-1}^6 s^{2-j} |D_t^{\frac{j}{4}}u|^2 + |\nabla u|^2 + s^{-2} \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) e^{2s\varphi_2} dxdt \\ &\geq \int_{D_3} \left(\sum_{j=-1}^6 s^{2-j} |D_t^{\frac{j}{4}}u|^2 + |\nabla u|^2 + s^{-2} \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) e^{2s\varphi_2} dxdt \\ &\geq e^{2s\mu_3} \int_{t_0 - \frac{\epsilon}{\sqrt{N}}}^{t_0 + \frac{\epsilon}{\sqrt{N}}} \int_{\Omega_0} \left(\sum_{j=-1}^6 s^{2-j} |D_t^{\frac{j}{4}}u|^2 + |\nabla u|^2 + s^{-2} \sum_{i,j=1}^n |\partial_i \partial_j u|^2 \right) dxdt. \end{aligned}$$

Therefore (45) yields

$$\|u\|_{H^{2,1}(\Omega_0 \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}))}^2 \leq Cs^3 e^{-2s(\mu_3 - \mu_2)} M^2 + Cs^2 e^{Cs} \mathcal{D}^2$$

for all $s \geq s_0$. Since $\sup_{s \geq s_0} s^3 e^{-s(\mu_3 - \mu_2)} < \infty$ and $s^2 \leq e^{Cs}$ for s large enough (e.g. $s \geq s_1$), we have

$$\|u\|_{H^{2,1}(\Omega_0 \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}))}^2 \leq Ce^{-C_0 s} M^2 + Ce^{Cs} \mathcal{D}^2$$

for all $s \geq s_2 := \max\{s_0, s_1\}$, $C_0 := \mu_3 - \mu_2 > 0$. After the change of the variable $s - s_2 \rightarrow s$, we obtain

$$\|u\|_{H^{2,1}(\Omega_0 \times (t_0 - \frac{\epsilon}{\sqrt{N}}, t_0 + \frac{\epsilon}{\sqrt{N}}))}^2 \leq Ce^{-C_0 s} M^2 + Ce^{Cs} \mathcal{D}^2 \quad (46)$$

for all $s \geq 0$. The generic constant C depends on s_2 , ϵ and the choice of Ω_0 , etc., but is still independent of s .

Let $m \in \mathbb{N}$ satisfy

$$\sqrt{2}\epsilon + \frac{m\epsilon}{\sqrt{N}} \leq T - \sqrt{2}\epsilon < \sqrt{2}\epsilon + \frac{(m+1)\epsilon}{\sqrt{N}} < T.$$

We here notice that the constant C is also independent of t_0 provided that $t_0 \in [\sqrt{2}\epsilon, T - \sqrt{2}\epsilon]$. Thus, by taking $t_0 = \sqrt{2}\epsilon + \frac{k\epsilon}{\sqrt{N}}$, $k = 0, 1, 2, \dots, m$ in (46), summing up over k and replacing $\sqrt{2}\epsilon$ by ϵ , we obtain

$$\|u\|_{H^{2,1}(\Omega_0 \times (\epsilon, T - \epsilon))}^2 \leq Ce^{-C_0 s} M^2 + Ce^{Cs} \mathcal{D}^2 \quad (47)$$

for all $s \geq 0$.

Finally, we show the estimate of Hölder type.

Firstly, let $\mathcal{D} = 0$. Then letting $s \rightarrow \infty$ in (47), we see that $u = 0$ in $\Omega_0 \times (\epsilon, T - \epsilon)$. So that the conclusion holds true.

Secondly, let $\mathcal{D} \neq 0$. We divide it into two cases. In the case of $\mathcal{D} \geq M$, we see that (47) implies $\|u\|_{H^{2,1}(\Omega_0 \times (\epsilon, T - \epsilon))} \leq Ce^{Cs}\mathcal{D}$ for all $s \geq 0$. This has already proved the theorem. Now if $\mathcal{D} \leq M$. We choose $s > 0$ minimizing the right-hand side of (47), that is,

$$e^{-C_0s}M^2 = e^{Cs}\mathcal{D}^2,$$

which yields

$$s = \frac{2}{C + C_0} \log \frac{M}{\mathcal{D}}$$

in view of that $\mathcal{D} \neq 0$. Then (47) gives

$$\|u\|_{H^{2,1}(\Omega_0 \times (\epsilon, T - \epsilon))}^2 \leq 2CM^{\frac{C}{C+C_0}}\mathcal{D}^{\frac{C_0}{C+C_0}}.$$

The proof of Theorem 4 is completed. \square

3.3 Application to an inverse source problem for the sup-diffusion

In this subsection, we consider another application of Theorem 3, say, the stability for the inverse source problem (Problem 1). We point out here that the uniqueness result for this kind of inverse problem can be proved by a similar argument in [7].

Before giving the proof of Theorem 5, we first recall the notation of the bounded domain Ω_1 defined in (3), and function $d \in C^2(\overline{\Omega_1})$ can be chosen so that (4) holds. Next for any $\epsilon > 0$, we recall the level set of ϵ related to function d as follows

$$\Omega_\epsilon := \{x \in \Omega : d(x) > \epsilon\}.$$

Note that $\Omega_\epsilon \neq \emptyset$ for small ϵ and $\overline{\Omega_\epsilon} \cap \partial\Omega \subset \Gamma$. Now we are ready to give a proof of the stability result.

Proof of Theorem 5. Recall that our weight function is chosen as

$$\varphi_2(x, t) = e^{\lambda\psi_2(x, t)}, \quad \psi_2(x, t) = d(x) - \beta(t - t_0)^2 + c_0.$$

Here, c_0 is a constant such that ψ_2 is always nonnegative and s, λ are two large parameters while the parameter $\beta > 0$ will be fixed later. Then on both sides of our governing equation (13), we integrate over the domain $\Omega_{3\epsilon}$ at time $t = t_0$ and obtain

$$\begin{aligned} \int_{\Omega_{3\epsilon}} |R(x, t_0)f(x)|^2 e^{2s\varphi_2(x, t_0)} dx &\leq \int_{\Omega_{3\epsilon}} |\partial_t u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx + C \int_{\Omega_{3\epsilon}} \left| D_t^{\frac{3}{4}} u(x, t_0) \right|^2 e^{2s\varphi_2(x, t_0)} dx \\ &\quad + \int_{\Omega_{3\epsilon}} |-\Delta u(x, t_0) + B(x) \cdot \nabla u(x, t_0) + c(x)u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx. \end{aligned} \quad (48)$$

Easily, we see that the third term on the RHS is bounded by $Ce^{Cs}\|u(\cdot, t_0)\|_{H^2(\Omega_{3\epsilon})}^2$ and the LHS can be estimated with some $C_0 > 0$:

$$\int_{\Omega_{3\epsilon}} |R(x, t_0)f(x)|^2 e^{2s\varphi_2(x, t_0)} dx \geq C_0 \int_{\Omega_{3\epsilon}} |f(x)|^2 e^{2s\varphi_2(x, t_0)} dx \quad (49)$$

under the first assumption in (14). The key point is how to estimate the first and second terms on the RHS of (48).

By our notation $\Omega_\epsilon = \{x \in \Omega : d(x) > \epsilon\}$ for any $\epsilon > 0$, we further set

$$Q_\epsilon := \{(x, t) \in Q : \psi_2(x, t) > \epsilon + c_0\}, \quad \epsilon > 0.$$

Then we have the following relations:

(i) $Q_\epsilon \subset \Omega_\epsilon \times (0, T)$,

(ii) $Q_\epsilon \supset \Omega_\epsilon \times \{t_0\}$.

In fact, if $(x, t) \in Q_\epsilon$, we have $d(x) - \beta(t - t_0)^2 > \epsilon$, i.e. $d(x) > \beta(t - t_0)^2 + \epsilon > \epsilon$. This means $x \in \Omega_\epsilon$. (i) is verified. On the other hand, if $x \in \Omega_\epsilon$ and $t = t_0$ then $\psi_2(x, t) = d(x) - \beta(t - t_0)^2 + c_0 = d(x) + c_0 > \epsilon + c_0$. That is, $(x, t) \in Q_\epsilon$. (ii) is verified. Furthermore, we choose $\beta = \frac{\|d\|_{C(\overline{\Omega_1})}}{\delta^2}$ where $\delta := \min\{t_0, T - t_0\}$ so that

(iii) $\overline{Q_\epsilon} \cap (\Omega \times \{0, T\}) = \emptyset$

is valid. Indeed, for $\forall (x, t) \in \Omega \times \{0, T\}$, $\psi(x, t) = d(x) - \beta(t - t_0)^2 + c_0 \leq \|d\|_{C(\overline{\Omega_1})} - \beta\delta^2 + c_0 = c_0$. This leads to $(x, t) \notin \overline{Q_\epsilon}$.

Relations (i) – (iii) guarantee that Q_ϵ is a sub-domain of Q and $\partial Q_\epsilon \cap \partial Q \subset \Gamma \times (0, T)$. Moreover, we assert that

(iv) $\Omega_{3\epsilon} \times (t_0 - \delta_\epsilon, t_0) \subset Q_{2\epsilon}$, $\delta_\epsilon := \sqrt{\frac{\epsilon}{\beta}} = \sqrt{\frac{\epsilon}{\|d\|}}\delta$.

Actually, for any $(x, t) \in \Omega_{3\epsilon} \times (t_0 - \delta_\epsilon, t_0)$, we have

$$\psi(x, t) = d(x) - \beta(t - t_0)^2 + c_0 > 3\epsilon - \beta\delta_\epsilon^2 + c_0 = 2\epsilon + c_0.$$

That is, $(x, t) \in Q_{2\epsilon}$.

Now we construct a function $\eta \in C^2[0, T]$ such that $0 \leq \eta \leq 1$ and

$$\eta = \begin{cases} 1 & \text{in } [t_0 - \frac{1}{2}\delta_\epsilon, t_0 + \frac{1}{2}\delta_\epsilon], \\ 0 & \text{in } [0, t_0 - \delta_\epsilon] \cup [t_0 + \delta_\epsilon, T] \end{cases}$$

for any small $\epsilon < \|d\|_{C(\overline{\Omega_1})}$. Then by the use of the condition (15), and noting that $\eta(t_0 - \delta_\epsilon) = 0$, $\eta(t_0) = 1$, we have

$$\begin{aligned} \int_{\Omega_{3\epsilon}} |\partial_t u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx &= \int_{\Omega_{3\epsilon}} |\eta(t_0) \partial_t u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx \\ &= \int_{t_0 - \delta_\epsilon}^{t_0} \frac{d}{dt} \int_{\Omega_{3\epsilon}} |\eta \partial_t u|^2 e^{2s\varphi_2} dx dt \\ &= \int_{t_0 - \delta_\epsilon}^{t_0} \int_{\Omega_{3\epsilon}} 2\eta \partial_t u (\eta \partial_t^2 u + \partial_t \eta \partial_t u) e^{2s\varphi_2} dx dt \\ &\quad - \int_{t_0 - \delta_\epsilon}^{t_0} \int_{\Omega_{3\epsilon}} 4(t - t_0) \beta s \lambda \varphi_2 |\eta \partial_t u|^2 e^{2s\varphi_2} dx dt \\ &\leq C \int_{Q_{2\epsilon}} (s^{-2} |\partial_t^2 u|^2 + s^2 |\partial_t u|^2) e^{2s\varphi_2} dx dt \end{aligned} \quad (50)$$

and

$$\begin{aligned} \int_{\Omega_{3\epsilon}} |D_t^{\frac{3}{4}} u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx &= \int_{\Omega_{3\epsilon}} |\eta(t_0) D_t^{\frac{3}{4}} u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx \\ &= \int_{t_0 - \delta_\epsilon}^{t_0} \frac{d}{dt} \int_{\Omega_{3\epsilon}} |\eta D_t^{\frac{3}{4}} u|^2 e^{2s\varphi_2} dx dt \\ &= \int_{t_0 - \delta_\epsilon}^{t_0} \int_{\Omega_{3\epsilon}} 2\eta D_t^{\frac{3}{4}} u (\eta \partial_t (D_t^{\frac{3}{4}} u) + \partial_t \eta D_t^{\frac{3}{4}} u) e^{2s\varphi_2} dx dt \\ &\quad - \int_{t_0 - \delta_\epsilon}^{t_0} \int_{\Omega_{3\epsilon}} 4\beta s \lambda \varphi_2 (t - t_0) |\eta D_t^{\frac{3}{4}} u|^2 e^{2s\varphi_2} dx dt \\ &\leq \frac{C}{s} \int_{Q_{2\epsilon}} (s^{-1} |D_t^{\frac{7}{4}} u|^2 + s^3 |D_t^{\frac{3}{4}} u|^2) e^{2s\varphi_2} dx dt. \end{aligned} \quad (51)$$

Now we intend to use the Carleman estimate established in Section 3.1 to evaluate the RHS of (50) and (51). For this, we take time derivative on both sides of (13) and then $u_1 := \partial_t u$ reads the following equation

$$\partial_t u_1 + q(x) \partial_t^{\frac{3}{4}} u_1 - \Delta u_1 + B \cdot \nabla u_1 + c u_1 = (\partial_t R) f, \quad \text{in } Q. \quad (52)$$

Noting the compatible condition $R(\cdot, 0) = 0$, we apply Theorem 3 to (52), and then we have

$$\begin{aligned} & \int_Q \chi_0^2 (s^{-2} |\partial_t u_1|^2 + s^{-1} |D_t^{\frac{3}{4}} u_1|^2 + s^2 |u_1|^2 + s^3 |D_t^{-\frac{1}{4}} u_1|^2) e^{2s\varphi_2} dx dt \\ & \leq C \int_Q \chi_0^2 \left(\sum_{j=3}^6 \left| D_t^{\frac{j}{4}} R \right|^2 \right) |f|^2 e^{2s\varphi_2} dx dt + Low1 + Bdy1 \end{aligned}$$

for all $s \geq s_1 \geq 1$, where the terms $Low1$ and $Bdy1$ are defined as

$$\begin{aligned} Low1 &= C s \int_Q \left(|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + \sum_{i,j=1}^n |\partial_i \partial_j \chi_0|^2 \right) \left(\sum_{j=-1}^2 (|\nabla (D_t^{\frac{j}{4}} u_1)|^2 + |D_t^{\frac{j}{4}} u_1|^2) \right) e^{2s\varphi_2} dx dt, \\ Bdy1 &= C e^{Cs} \int_{\partial Q} (|\partial_t \chi_0|^2 + |\nabla \chi_0|^2 + |\chi_0|^2) \sum_{j=-1}^2 (|\nabla_{x,t} (D_t^{\frac{j}{4}} u_1)|^2 + |D_t^{\frac{j}{4}} u_1|^2) dS dt. \end{aligned}$$

We choose the cut-off function $\chi_0 \in C^\infty(\mathbb{R}^{n+1})$ such that $0 \leq \chi_0 \leq 1$ and

$$\chi_0 = \begin{cases} 1, & \psi_2(x, t) > 2\epsilon + c_0, \\ 0, & \psi_2(x, t) \leq \epsilon + c_0. \end{cases}$$

Therefore $\chi_0 = 1$ in $Q_{2\epsilon}$ while its derivatives vanish in $Q_{2\epsilon}$ which enable us to rewrite the above Carleman inequality as follows:

$$\begin{aligned} & \int_{Q_{2\epsilon}} (s^{-2} |\partial_t^2 u|^2 + s^{-1} |D_t^{\frac{7}{4}} u|^2 + s^2 |\partial_t u|^2 + s^3 |D_t^{\frac{3}{4}} u|^2) e^{2s\varphi_2} dx dt \\ & \leq C \int_{Q_\epsilon} \left(\sum_{j=3}^6 \left| D_t^{\frac{j}{4}} R \right|^2 \right) |f|^2 e^{2s\varphi_2} dx dt + Low2 + Bdy2 \end{aligned} \tag{53}$$

for all $s \geq s_1 \geq 1$, where

$$\begin{aligned} Low2 &= C s e^{2s\epsilon^{\lambda(2\epsilon+c_0)}} \int_{Q_\epsilon \setminus Q_{2\epsilon}} (|\nabla (D_t^{\frac{3}{2}} u)|^2 + |D_t^{\frac{3}{2}} u|^2) dx dt, \\ Bdy2 &= C e^{Cs} \int_{\Gamma \times (0, T)} (|\nabla_{x,t} (D_t^{\frac{3}{2}} u)|^2 + |D_t^{\frac{3}{2}} u|^2) dS dt. \end{aligned}$$

The above inequality is derived from the choice of χ_0 and the estimate (40) as well as the relations (i)-(iii) which indicate that both the derivatives of χ_0 and χ_0 itself vanish on ∂Q except for the lateral boundary $\Gamma \times (0, T)$. Now substituting (53) into (50) and (51) implies that

$$\begin{aligned} & \int_{\Omega_{3\epsilon}} |\partial_t u(x, t_0)|^2 e^{2s\varphi_2(x, t_0)} dx + \int_{\Omega_{3\epsilon}} \left| D_t^{\frac{3}{4}} u(x, t_0) \right|^2 e^{2s\varphi_2(x, t_0)} dx \\ & \leq C \int_{Q_\epsilon} \left(\sum_{j=3}^6 \left| D_t^{\frac{j}{4}} R \right|^2 \right) |f|^2 e^{2s\varphi_2} dx dt + Low2 + Bdy2. \end{aligned}$$

Combined with (48) and (49), we obtain

$$\int_{\Omega_{3\epsilon}} |f(x)|^2 e^{2s\varphi_2(x, t_0)} dx \leq C e^{Cs} \|u(\cdot, t_0)\|_{H^2(\Omega_{3\epsilon})}^2 + C \int_{Q_\epsilon} |f|^2 e^{2s\varphi_2} dx dt + Low2 + Bdy2.$$

Moreover, we divide the second term on the RHS into two parts:

$$C \int_{Q_\epsilon} |f|^2 e^{2s\varphi_2} dx dt = C \int_{Q_{3\epsilon}} |f|^2 e^{2s\varphi_2} dx dt + C \int_{Q_\epsilon \setminus Q_{3\epsilon}} |f|^2 e^{2s\varphi_2} dx dt$$

$$\leq C \int_{Q_{3\epsilon}} |f|^2 e^{2s\varphi_2} dx dt + C e^{2se^{\lambda(3\epsilon+c_0)}} \int_{Q_\epsilon \setminus Q_{3\epsilon}} |f|^2 dx dt,$$

which leads to

$$\begin{aligned} \int_{\Omega_{3\epsilon}} |f(x)|^2 e^{2s\varphi_2(x,t_0)} dx &\leq C \int_{Q_{3\epsilon}} |f|^2 e^{2s\varphi_2} dx dt + C e^{2se^{\lambda(3\epsilon+c_0)}} \int_{Q_\epsilon \setminus Q_{3\epsilon}} |f|^2 dx dt \\ &\quad + C e^{2se^{\lambda(2\epsilon+c_0)}} \int_{Q_\epsilon} (|\nabla(D_t^{\frac{3}{2}} u)|^2 + |D_t^{\frac{3}{2}} u|^2) dx dt + C e^{Cs} \mathcal{D}^2 \\ &\leq C \int_{Q_{3\epsilon}} |f|^2 e^{2s\varphi_2} dx dt + C e^{2se^{\lambda(3\epsilon+c_0)}} M^2 + C e^{Cs} \mathcal{D}^2 \end{aligned}$$

for all $s \geq s_1 \geq 1$. Here M and \mathcal{D} denote a priori bound and measurements defined in Theorem 5. Since $\varphi_2(x, t)$ attains its maximum at $t = t_0$, we can absorb the first term on the RHS into the LHS by taking s large enough (e.g. $s \geq s_2$). That is

$$\int_{\Omega_{3\epsilon}} |f(x)|^2 e^{2s\varphi_2(x,t_0)} dx \leq C s e^{2se^{\lambda(3\epsilon+c_0)}} M^2 + C e^{Cs} \mathcal{D}^2$$

for all $s \geq s_3 = \max\{s_1, s_2\}$. On the hand hand,

$$\int_{\Omega_{3\epsilon}} |f(x)|^2 e^{2s\varphi_2(x,t_0)} dx \geq \int_{\Omega_{4\epsilon}} |f(x)|^2 e^{2s\varphi_2(x,t_0)} dx \geq e^{2se^{\lambda(4\epsilon+c_0)}} \|f\|_{L^2(\Omega_{4\epsilon})}^2.$$

Therefore, we obtain

$$\|f\|_{L^2(\Omega_{4\epsilon})}^2 \leq C s e^{-2\epsilon_0 s} M^2 + C e^{Cs} \mathcal{D}^2$$

for all $s \geq s_3$. Here $\epsilon_0 := e^{\lambda(4\epsilon+c_0)} - e^{\lambda(3\epsilon+c_0)} = e^{\lambda(3\epsilon+c_0)}(e^{\lambda\epsilon} - 1) > 0$. Since $\sup_{s>0} s e^{-\epsilon_0 s} < \infty$, the above inequality gives

$$\|f\|_{L^2(\Omega_{4\epsilon})}^2 \leq C e^{-\epsilon_0 s} M^2 + C e^{Cs} \mathcal{D}^2 \quad (54)$$

for all $s \geq \hat{s}$ with $\hat{s} \geq s_3$ satisfying $\hat{s} e^{-\epsilon_0 \hat{s}} \leq C$. By substituting s by $s + \hat{s}$, inequality (54) holds for all $s \geq 0$ with a larger generic constant $C e^{\hat{s}}$ which is again denoted by C .

Finally, we repeat the argument in the proof of Theorem 4 to show the estimate of Hölder type:

$$\|f\|_{L^2(\Omega_{4\epsilon})} \leq C(M^{1-\theta} \mathcal{D}^\theta + \mathcal{D}).$$

The proof of Theorem 5 is completed. \square

4 Conclusions and remarks

In this paper, we considered the Carleman estimates for the time-fractional advection-diffusion equation (2) and the applications.

First, in the case of sub-diffusion, that is, the largest fractional order is strictly less than half, the Carleman estimate for the equation (2) was established by regarding the fractional order terms as perturbation of the first order time-derivative and the use of the Carleman estimate for the parabolic equations. As an application, the conditional stability for a lateral Cauchy problem was obtained, say, the solution of the equation (2) is continuously dependent of not only the partial Cauchy data and the source term but also the initial value. Due to our choice of the weight function $\psi_1(x, t) = d(x) - \beta t^{2-2\alpha_1}$, we do not know whether the estimate is valid without the initial value, and this remains open. On the other hand, the choice of the new weight function ψ_1 is not suitable for the study of the inverse problems. As is well known, for dealing with the inverse problems, the Carleman type estimate derived by $d(x) - \beta(t - t_0)^2 + c_0$ ($t_0 \in (0, T)$) should be better according to the series of theories in [18]. The inverse problems for the equation (2) in the case of $\alpha_1 < \frac{1}{2}$ remain open. The above arguments cannot deal with the case of sup-diffusion where the largest order is greater than half neither. However, in the case of the largest order is rational number and less than $\frac{3}{4}$, we found the Carleman estimate with a cut-off function for the equation (7) by using the regular weight

function $d(x) - \beta(t - t_0)^2 + c_0$ ($t_0 \in (0, T)$) can be constructed. Then by an argument similar to that in [18], the conditional stability for the inverse source problem (Problem 1) was proved as well as the conditional stability for the lateral Cauchy problem. The fractional order $\alpha = \frac{3}{4}$ is the largest one which one can deal with based on our arguments of Carleman estimate. Moreover, constant \hat{s} and the generic constant C and in Theorem 3 depend on the denominator of fractional order α which prevents us from extending the order to real number by the density of \mathbb{Q} in \mathbb{R} . The case of the general order remains open.

Finally, it should be mentioned that the stability inequalities in Theorem 4 immediately gives the unique continuation result of (7), i.e. the solution of the equation (7) must vanish in the whole domain Q if its initial value is identically 0 in Ω and the partial boundary data $u|_{\Gamma \times (0, T)}$ and $\partial_\nu u|_{\Gamma \times (0, T)}$ are zero. This principle is called a weak type unique continuation for the equation (2) since the homogeneous initial value is not essential for the unique continuation (UC) (e.g., UC for the elliptic equations or UC for the parabolic equations). However, we cannot repeat this argument to derive the weak unique continuation of (2) because the constant $\varepsilon > 0$ depends on the choice of Ω_0 in the estimate in Theorem 2. We refer to [7] and [15] for other kind of weak type unique continuation where the initial value does not vanish but the homogeneous Dirichlet or Neumann boundary condition is required.

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